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Market Valuation of Illiquid Debt and Implications for Conflicts Among Creditors

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Abstract

We develop a formula for the market value of debt when the borrower's repayment capacity varies stochastically, and shortfalls are rolled over. The value of a marginal dollar of nominal claim is an S-shaped function of the ratio of the repayment capacity to the amount of nominal debt. Shifts of this curve are examined in response to changes in the underlying parameters. The calculations bring out some conflicts of interest among lenders of differing degrees of seniority. Most surprisingly, junior creditors gain when the loan is rescheduled on terms more favorable to the debtor.

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Summary

This paper focuses on inter-creditor conflicts. It defines debt-servicing difficulties as being broadly of two types, one stemming from the country's inability to pay (illiquidity) and the other from its unwillingness to pay (strategic default).

Consider a country with a given amount of outstanding debt with preset terms, or contractual servicing obligation. The country meets its contractual obligation within the limits of its ability to do so, and makes payments to creditors in order of the seniority of their claims. This order may be a part of the contract, or a de facto arrangement; for example, it is often thought that official creditors' claims get priority over commercial banks' claims even though there is no formal seniority. Arrears in payments are rolled over at the preset terms, and all rollovers of an unpaid claim remain senior to all the claims that originally ranked junior to it, and their rollovers.

If the terms of the loan are altered to become more favorable to the country, then the market value of relatively senior claims falls, but that of relatively junior claims increases. On the other hand, if the volatility of the country's ability to pay increases, then the market value of senior claims goes down and that of junior claims goes up. To the extent that such volatility can be affected by the country's policies, therefore, senior creditors will advise the country to play it safe, while junior creditors will favor risky policies.

The technique developed in this paper has the potential for addressing other questions such as debt-equity swaps and burden-sharing. These are suggested as topics for further research.

I. Introduction

When a country experiences difficulty in servicing its debt, conflicts of interest arise between the country and its creditors, and also among creditors who hold claims of different standing. Analytical treatments of country debt mostly deal with games of conflict between debtors and creditors. In Eaton, Gersovitz and Stiglitz (1986), Bulow and Rogoff (1989) etc., the debtor country strategically defaults on its obligation whenever it stands to gain economically from so doing, given the credible punishments its creditors can inflict.

Conflicts of interest among different creditors are at least as relevant in practice, but are relatively neglected in the theoretical literature. Creditors holding different types of claims, or debt of different degrees of seniority, are differently affected by various shifts in the underlying parameters of the situation. Therefore they would react differently to proposals to renegotiate the debt on different terms, and would recommend different policies to the country. This paper contributes to our understanding of such conflicts.

Inter-creditor conflicts can arise even without the complication of strategic default by the creditor. The debtor's illiquidity, or inability to pay, can affect creditors just as much as strategic default, or unwillingness to pay. To keep our analysis simpler, and to counterbalance the attention that has been devoted to strategic default, we consider only illiquidity as the reason for the debt problem. Some would argue that it is the more important issue in reality anyway.

The attitude of a creditor to debt renegotiation or policy changes will be governed by the effect on the value of his claim. Therefore, the analysis must begin by finding the market values of such claims. We focus on debt with the following characteristics. The claim is serviced at a fixed coupon rate, which exceeds the riskless rate of interest because of the possibility of illiquidity. Second, the country's debt service capacity fluctuates over time in an exogenous but stochastic manner. Repayment occurs within this limit and up to the contractual amount, in order of seniority. Any shortfall below the contractual payment is rolled over at the same coupon rate. Third, if two claimants have different seniorities and neither gets paid at a point in time, then all claims of the more senior creditor, the original one as well as the new one arising from the rollover, are senior to all of the more junior creditor's claims. Creditors of equal standing remain equal in their old and new claims. We do not consider the issue of the new debt, but in this framework it would be easy to do so provided new debt is junior to all existing debt.

These specifications seem reasonable first approximations, but future work should consider alternatives. In particular, some magnitudes that we take as exogenous should be endogenized. For example, we consider only debt that was contracted in the past, and take its coupon rate as exogenous, posing the question of how its value changes when some unexpected change in the underlying circumstances occurs. But when a new debt contract is being considered, its terms should be determined endogenously, assuming rational expectations about the stochastic future. The process for the evolution of the debt-servicing capacity can also be endogenous, with possible feedback from the level of the debt itself; this will capture the idea that a country's investment is affected by its debt-service obligation.

Some previous analyses of related issues exist in the literature. Dooley (1989) and Helpman (1989) come closest to our concerns. They stipulate a probability distribution on the discounted present value of the country's ability to repay. This reduces the problem to two periods, and conceals the dynamics of valuation and renegotiation. Claessens and Wijnbergen (1989) and Cohen (1990) come closest to our techniques, but make several unrealistic assumptions to get simple solutions. Claessens and Wijnbergen assume that any arrears in repayment are simply forgiven. This enables them to cast the model and the results in terms of textbook Black-Scholes option pricing formulas, but reality lies much closer to the extreme of full rollover of arrears than that of total forgiveness. Cohen assumes that all of the debt-servicing capacity must be paid over at all times. When this amount exceed the contractual obligation, the excess must be used for early retirement of the debt. This does not conform to the behavior of debtor countries. Cohen must also assume that the coupon rate on these contracts equals the riskless interest rate notwithstanding the illiquidity risk; this the only case where his differential equation governing value has a solution in terms of simple functions. We offer better assumptions in all these respects. Of course, the solution becomes harder. Finally, our focus on the variation of values by the seniority of the claim, and the implications for conflicts of interest among creditors, are substantive new features.

Bartolini (1989) uses the same model as we do, and uses it to answer a different question, namely under what circumstances will new voluntary lending occur.

We find that the market value of a marginal dollar's nominal claim can be expressed as an S-shaped function of the ratio of the country's debt-servicing capacity to the seniority rank of the claim. Thus, the value of a given claim rises when the servicing capacity increases, and for a given servicing capacity, more senior claims have higher value. This leads to an 'externality' among claims of different standing. When the arrears on a senior claim are rolled over, this decreases the probability of repayment for all the more junior claims, which reduces their market values. This in turn causes inter-creditor conflicts of interest. The calculations can also be used in a cooperative or normative manner, to decide how much of the arrears should be forgiven by creditors of different standing, with the aim of sharing the burden equitably in some specified sense; see Dooley (1990) for numerical examples bearing on this issue.

II. The Model

We consider a debtor with a stock of D dollars of nominal claims. Each claim entitles its holder to a flow of c dollars in perpetuity, subject to the debtor having sufficient repayment capacity. If this condition was expected never to bind, c would equal the riskless rate of interest r. Given the risk that the debtor may, from time to time, be unable to service the claim, c will exceed r. We do not consider the endogenous determination of c, but our model will be a useful input to the analysis of that question.

The capacity to service debt is denoted by X. When the debtor is a country, this should be thought of as its trade surplus. The evolution of X over time is a stochastic process, driven by exogenous shocks to exchange rates, resource prices, etc. For analytical simplicity, we suppose the process to be a geometric Brownian motion,

$$dX/X = \mu dt + \sigma dw, \tag{1}$$

where w is the standard Wiener process, with $E(dw) = 0$ and $E(dw^2) = dt$. We need $r > \mu$ to ensure that the expected present value of the repayment capacity is finite.

The Brownian motion specification has some claim to attention because exchange rate and resource price movements are often adequately described in this way. It also allows us to examine the consequences of changes in the trend parameter μ and the volatility parameter σ. But it has limitations, the most obvious ones being that X cannot be negative, and has no tendency to revert to a long-run average. The trend and volatility can also be endogenous, with feedback from other variables including D. Future work should consider such extensions.

The flow payment on the stock D of debt is cD. When X is greater than cD, the full amount due is paid. When X falls short of this, X is paid, the difference is rolled over, and added to the stock of nominal claims. Therefore, the dynamics of D is given by

$$dD = \begin{cases} 0 & \text{when } X \geq cD \\ (cD - X)dt & \text{when } X < cD. \end{cases} \tag{2}$$

By modifying the right-hand side, this can be generalized to allow forgiveness of a portion of arrears. In future work, the fraction forgiven can even be made endogenous, to achieve some specified goal of burden-sharing among creditors.

We assume that these claims can be traded in a competitive market with risk-neutral speculators. The value of the whole debt is then the expected discounted present value of all future receipts, namely

$$V(X,D) = E\left\{\int_0^\infty \min(X_t,\ cD_t)e^{-rt}dt \,\middle|\, (X_0,D_0) = (X,D)\right\}, \tag{3}$$

where X_t and D_t evolve according to (1) and (2). The task is to obtain various properties of the function V.

We begin by pointing out that the same procedure can be used for valuing the most senior D dollars of nominal claims out of a larger total stock of debt. Now, in writing (2), the assumption is that the amount rolled over from an unpaid senior claim is also senior to all the originally more junior claims and their rollovers. This seems realistic.

In the context of country debt, official lenders such as the IMF and the World Bank are often de facto senior to the private lenders (commercial banks). But there are large groups of lenders of equal seniority. Therefore, it is important that our method can handle this case, and it can. Consider a range of claims of nominal value D_2, equal in seniority among themselves, and ranking below the most senior D_1 (which may be zero) dollars of claims. Even though our group of claimants of size D_2 is of equal standing, we can rank the claims in an arbitrary order of seniority ranging from D_1 to (D_1+D_2), and then suppose that each claimant holds an equiproportionate mixture of all the claims so ranked. In other words, we find the total value of all the claims of equal standing, and then average it out over the holders of these claims. In symbols, each unit claim in this range has value $[V(X,D_1 + D_2) - V(X,D_1)]/D_2$.

Finally, if debt is the only contractual obligation, equity is the residual. The expected present value of the country's repayment capacity is simply $X/(r-\mu)$. Therefore, the value of all equity is

$$S(X,D) = X/(r-\mu) - V(X,D). \tag{4}$$

This can be the starting point for an analysis of debt-equity swaps in a debt-equity swaps in a dynamic context. We leave that as a topic for future research.

We relegate the technical details of the solution to an appendix, and present only the results and some economic intuition in the text. The first step in the solution process is to convert (3) into an arbitrage equation for the asset consisting of all the claims. Formally, this is a decomposition of the integral similar to that in Dynamic Programming. Starting at any point in time, over the next little interval dt, the asset earns a dividend equal to min(X,cD)dt. Also, as X and D change, the value changes, and this constitutes a capital gain or loss whose expected value is E[dV]. The arbitrage condition requires that the dividend and the expected capital gain add up to the normal return, or

$$\min(X,cD)dt + E[dV] = rV\,dt. \tag{5}$$

We can express dV in terms of dX and dD using a Taylor expansion, and then use (1) and (2) and take expectations. Since the expansion is to be substituted into (5), we need retain only terms of order dt. But we must remember that in (1) the expectation of dw^2, and hence that of dX^2, is of order dt. Therefore, the expansion of V with respect to X must be carried on to the second order. This is the essence of Itô's Lemma in this context.

Note also that dD behaves differently according to whether X is greater than or less than cD. When all this is done, we finally find that, for the 'liquid region' $X > cD$,

$$\frac{1}{2}\sigma^2 X^2 V_{XX} + \mu X V_X - rV + cD = 0, \tag{6}$$

and in the 'illiquid region' $X < cD$,

$$\frac{1}{2}\sigma^2 X^2 V_{XX} + \mu X V_X + (cD - X) V_D - rV + X = 0. \tag{7}$$

It is obvious from (1)-(3) that V is homogeneous of degree one in (X,D). Therefore, the average value $v = V(X,D)/D$, and the value of the marginal claim $m = V_D(X,D)$, are functions only of the ratio $x = X/D$. The solution is more easily obtained, and better understood, in terms of these functions. In particular, by examining the change in the value of the marginal claim as D changes, we can understand the differences in the interests of creditors of different seniority. Therefore, we express the equations in terms of the marginal value function m(x). We find that in the liquid region $x > c$,

$$\frac{1}{2}\sigma^2 x^2 m''(x) + \mu x m'(x) - rm(x) + c = 0, \tag{8}$$

while in the illiquid region $x < c$,

$$\frac{1}{2}\sigma^2 x^2 m''(x) + (\mu - c + x) x m'(x) - (r-c)m(x) = 0. \tag{9}$$

The solution is obtained in the appendix, and we merely state it here:

$$m(x) = \begin{cases} c/r - Ax^{-\alpha} & \text{for } x > c \\ B\, x^{\nu}\, H(-2x/\sigma^2,\ \nu, \omega) & \text{for } x < c. \end{cases} \tag{10}$$

The notation is as follows. Consider the quadratic equation.

$$q(\xi) \equiv \frac{1}{2}\xi(\xi - 1) + \mu\xi - r = 0.$$

This has one negative root, which we write as $-\alpha$. (The other root exceeds 1.) Next, the quadratic equation

$$Q(\xi) \equiv \frac{1}{2}\sigma^2 \xi(\xi-1) + (\mu-c)\xi - (r-c) = 0$$

has two roots, one between 0 and 1, and the other, which we write as v, greater than 1. Next,

$$\omega = 2\nu + 2(\mu-c)/\sigma^2,$$

and H is the confluent hypergeometric function,

$$H(z;\nu,\omega) = 1 + \frac{\nu}{\omega} z + \frac{1}{2!}\frac{\nu(\nu+1)}{\omega(\omega+1)} z^2 + \frac{1}{3!}\frac{\nu(\nu+1)(\nu+2)}{\omega(\omega+1)(\omega+2)} z^3 + \ldots \tag{11}$$

This function can be thought of as a generalization of the exponential, and has many similar properties. In fact, if we were to set ω and ν equal to each other, H would reduce to the exponential. Finally, the constants A and B are chosen to make the two branches of m(x) meet smoothly at x = c, satisfying the Value Matching Condition m(c-) = m(c+) and the Smooth Pasting Condition m'(c-) = m'(c+).

Once m(x) is known, v(x) can be obtained using the relationship

$$v(x) - xv'(x) = m(x). \tag{12}$$

Conversely, the appendix shows how v(x) can be obtained directly as the solution of a differential equation derived from (6) and (7).

Although we have an explicit solution for m(x), its properties are difficult to establish algebraically. Therefore, we obtained numerical solutions for a wide range of values of the parameters. We report a representative sample of these experiments, and discuss their economic implications.

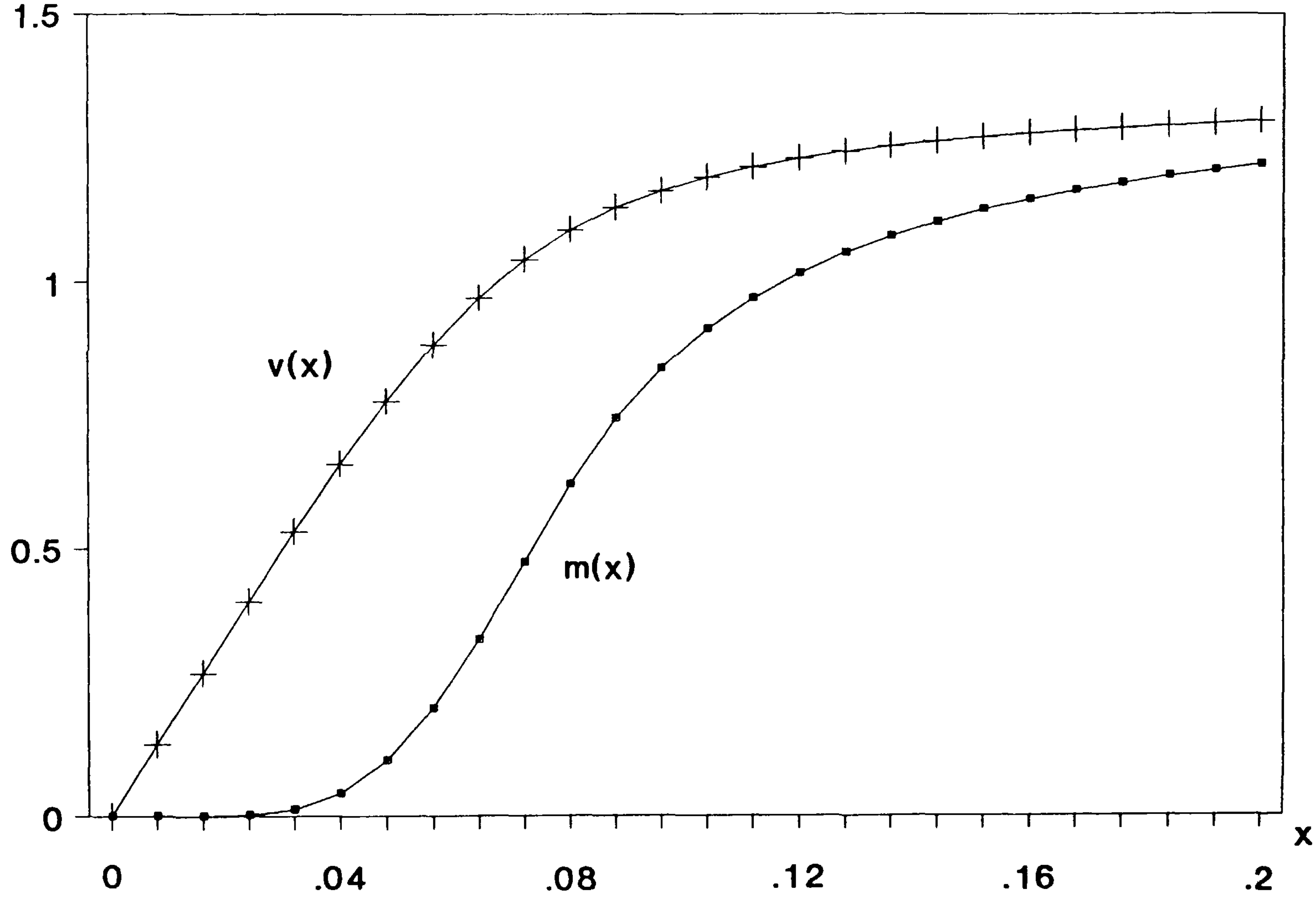
III. Solution and Implications

The typical solutions for m(x) and v(x) are shown in Figure 1. The choice of parameters follows Bartolini (1989). For the coupon rate c we chose the most recent average rate charged to highly indebted countries by private creditors, 8 percent. The riskless interest rate was a rounded down value of the LIBOR in 1987, namely, 6 percent. We chose the trend μ = 0.00, and the volatility parameter σ^2 = 0.02. The former is somewhat pessimistic, but the latter roughly conforms to the levels of variability of resource prices and exchange rates.

There are two ways of viewing the solutions. An increase in x = X/D can be thought of either as an increase in X holding D fixed, or a decrease in D holding X fixed. The first interpretation focuses on a claim of a given level of seniority, and asks how its value changes as the debt-servicing capacity increases. The second interpretation looks at a debtor

Figure 1.

$c = 0.08, \quad r = 0.06, \quad \mu = 0.00, \quad \sigma^2 = 0.02$



with a given debt-servicing capacity, and asks how the values of claims change as we vary the degree of seniority; higher x means greater seniority.

The solution for m(x) is S-shaped. It starts at 0 for x = 0, rises slowly at first, then faster, and slows down again to approach the asymptotic value c/r. Correspondingly, v(x) is concave, and approaches the same asymptote.

These results make good intuitive sense. The value of a given claim rises as the debtor's servicing capacity rises, and for a given servicing capacity, the value of a claim rises with its seniority. As x goes to ∞, that is when the servicing capacity is very large, or we are looking at the most senior claim, the value approaches the full capitalized value of the coupon.

The S-shape also has an intuitive interpretation using an analogy with financial options. The analogy is not exact because of the rollover feature of the debt claim, but a sufficiently useful parallel remains. A claim with very low x is 'far out of the money,' one near x = c is 'just in the money,' while one with very large x is 'deep in the money.' The value of a claim is most responsive to an improvement in the underlying fundamental when the claim is just in the money. Actually, the point where the slope of m(x) is steepest comes a little before x = c; this is a consequence of the rollover feature. When a claim is far out of the money, it responds to the fundamentals very little in absolute terms, but quite sensitively in percentage terms. In the same way, the elasticity of m(x) is highest, and approximately equal to the ν, when x is small, and gradually falls to zero as x increases to ∞.

For the parameter values of Figure 1, we have $\nu = 8.77$. Therefore, the elasticity of m(x) is very high when x is small. That is why m(x) curve starts out so flat and rises so slowly at first. As a result, claims that are significantly out of the money have very little value. If the debtor has only half of the ability that would be needed to service your claim, so $x = \frac{1}{2}c$ or 4 percent in Figure 1, we find that m(x) = 0.042. Since the full value is c/r = 1.33, this claim is worth only 4.2/1.33 = 3.15 cents on the dollar. By the time x has risen to c, that is, for a claim that is just on the verge of being serviced, the value has risen to 0.62, which is 46.6 cents on the dollar.

Because of this, the junior creditors are hurt most when the country's capacity to service debt declines stochastically. Intuitively, the rollover of the more senior creditors' claims makes the prospects of future payment of the junior claims even worse. In this sense, the burden of the country's bad luck falls disproportionately on the junior creditors. This was pointed out by Dooley (1990) using numerical examples. If the senior creditors are official lenders, they are presumably not solely concerned with the market value of their own claims, and open to a cooperative arrangement for more equitable burden-sharing. We can analyze this in our framework by supposing that a fraction $\delta(X/D)$ of a claim is rolled over, thus changing (2) to

$$dD = \begin{cases} 0 & \text{when } X \geq cD \\ \delta(X/D)(cD-X)dt & \text{when } X < cD, \end{cases}$$

and then choosing the function $\delta(X/D)$ to achieve a specified pattern of effect on the marginal values.

We referred to this solution as 'typical,' but a qualitatively different solution arises for certain parameter ranges, namely when μ is large or when σ is very small. Figure 2 shows such a case, when c is 5 percent, r is 4 percent, μ is 3 percent, and σ^2 is 1 percent. Here both v(x) and m(x) rise above c/r when x is somewhat below c, and then fall to approach c/r asymptotically.

One does not expect the market value of a claim to exceed the capitalized value of the coupon, so this kind of solution is quite counterintuitive. But it can be explained by considering the extreme case when $\sigma = 0$. Now, in the illiquid region $x < c$, the dynamics of x is governed by

$$dx/x = dX/X - dD/D = \mu - c + x.$$

If the starting point is $x_0 > c - \mu$, then x increases, and ultimately crosses c into the liquid region. Over this range the creditor is accumulating additional claims at the coupon rate, which exceeds the riskless interest rate. But there is no risk; the servicing capacity is sure to cross over into the liquid region. Then it is not surprising that the original claim can have a value in excess of c/r. While this is formally correct, one does not expect such a claim to exist in a more general equilibrium model of lending. The coupon rate will fall until such pure profit opportunities are eliminated. Therefore, we will not consider solutions of this kind any further.

Let us return to Figure 1. The shape of the m(x) curve has an implication for conflicts of interest among creditors. We can think of an increase in X as a shift in the debtor country's terms of trade. For example, it can result from a lowering of the industrial countries' trade barriers against the LDCs. Note that X will continue to follow its probabilistic law of motion from the new higher level. The industrial countries regard such trade liberalization as costly, and will undertake it only in response to pressure from potential beneficiaries. Among holders of LDC debt, who benefits the most from such a shift? The answer depends on whether the absolute or the relative change in the value of the claim is the relevant consideration. Original holders of the claims might be more concerned with the absolute increase in the value, whereas secondary holders who bought the claims at the market prices might want the rate or return, or

Figure 2.

$c = 0.05,\ r = 0.04,\ \mu = 0.03,\ \sigma^2 = 0.01$

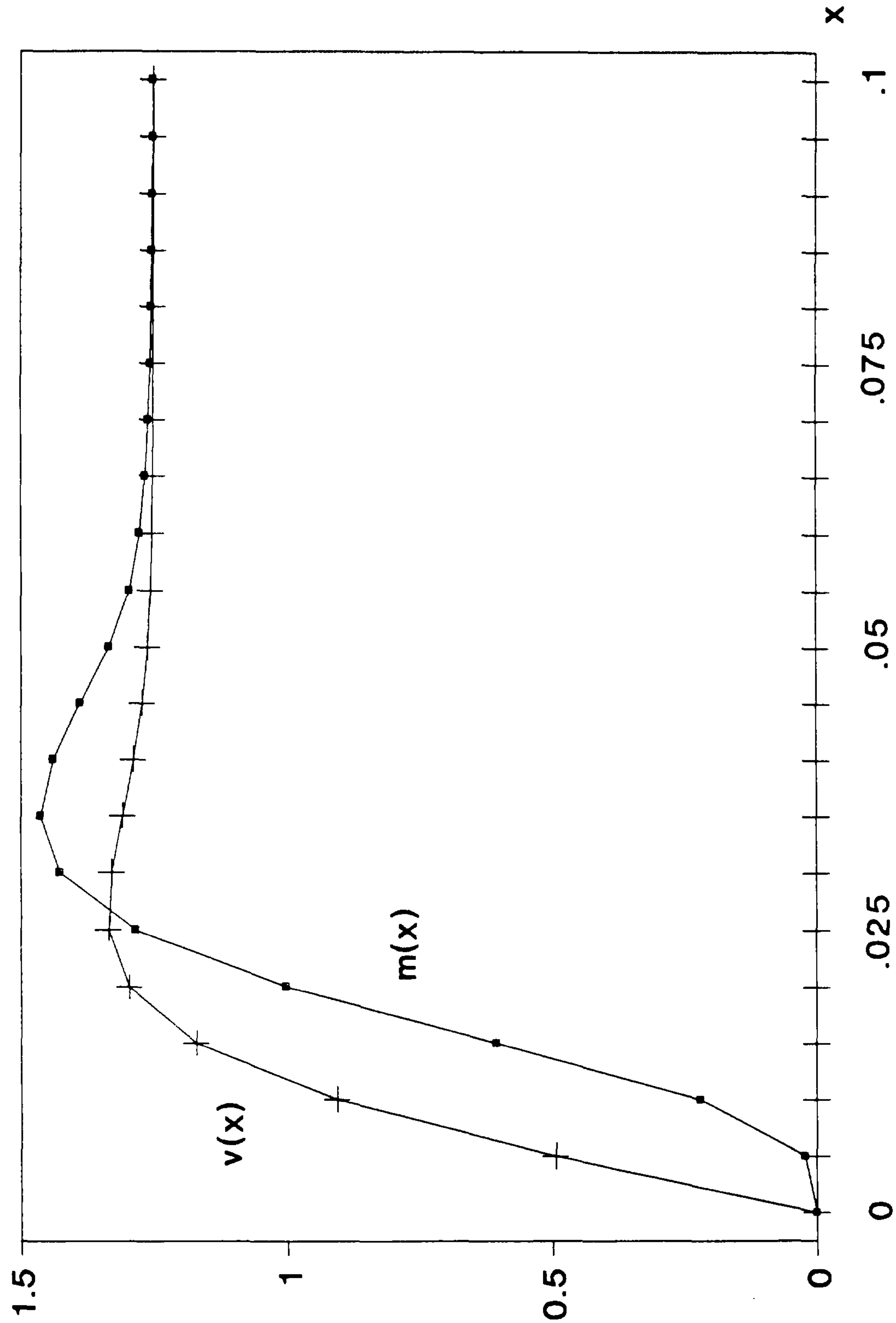


Figure 3.

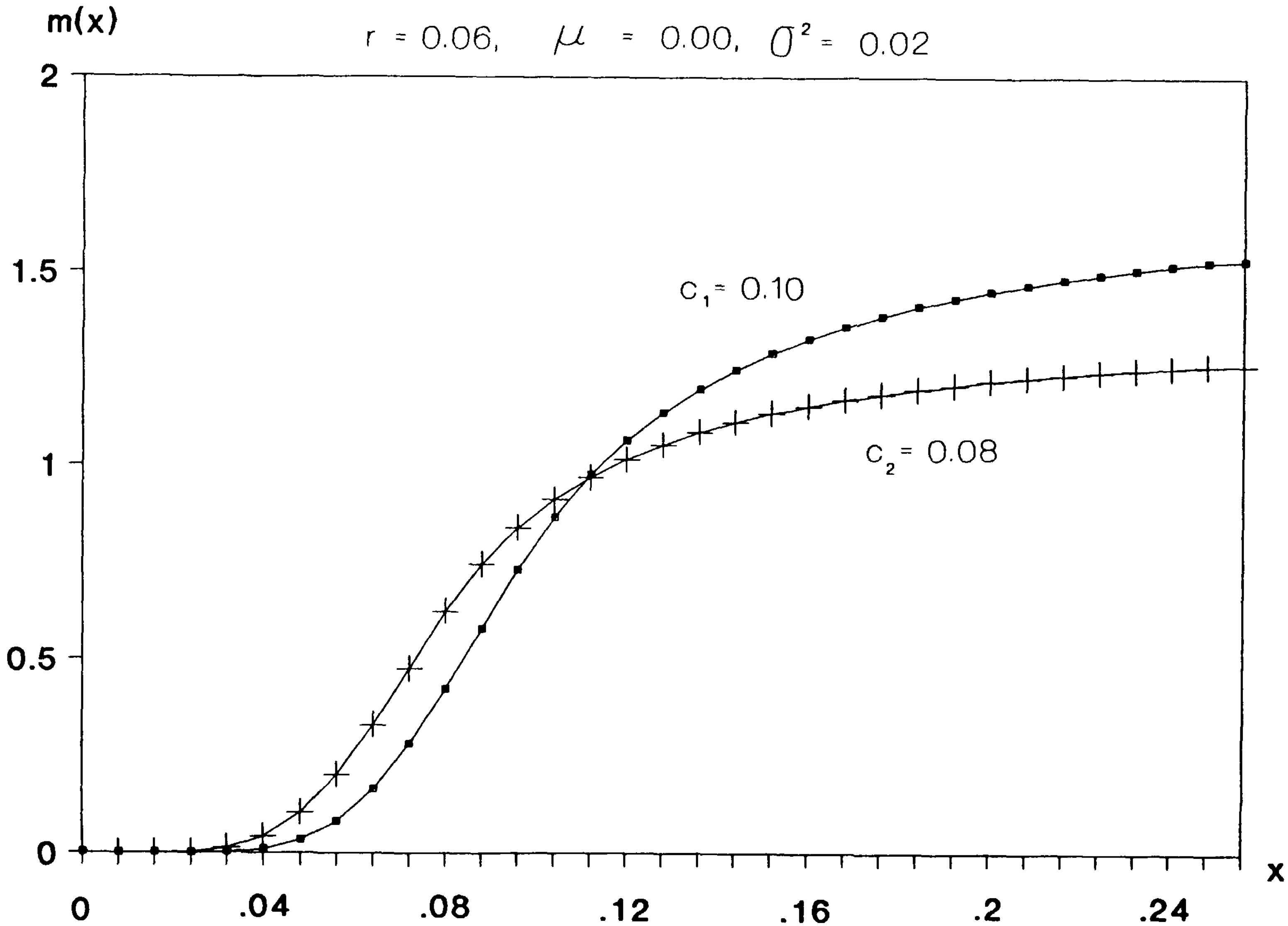


Figure 4.

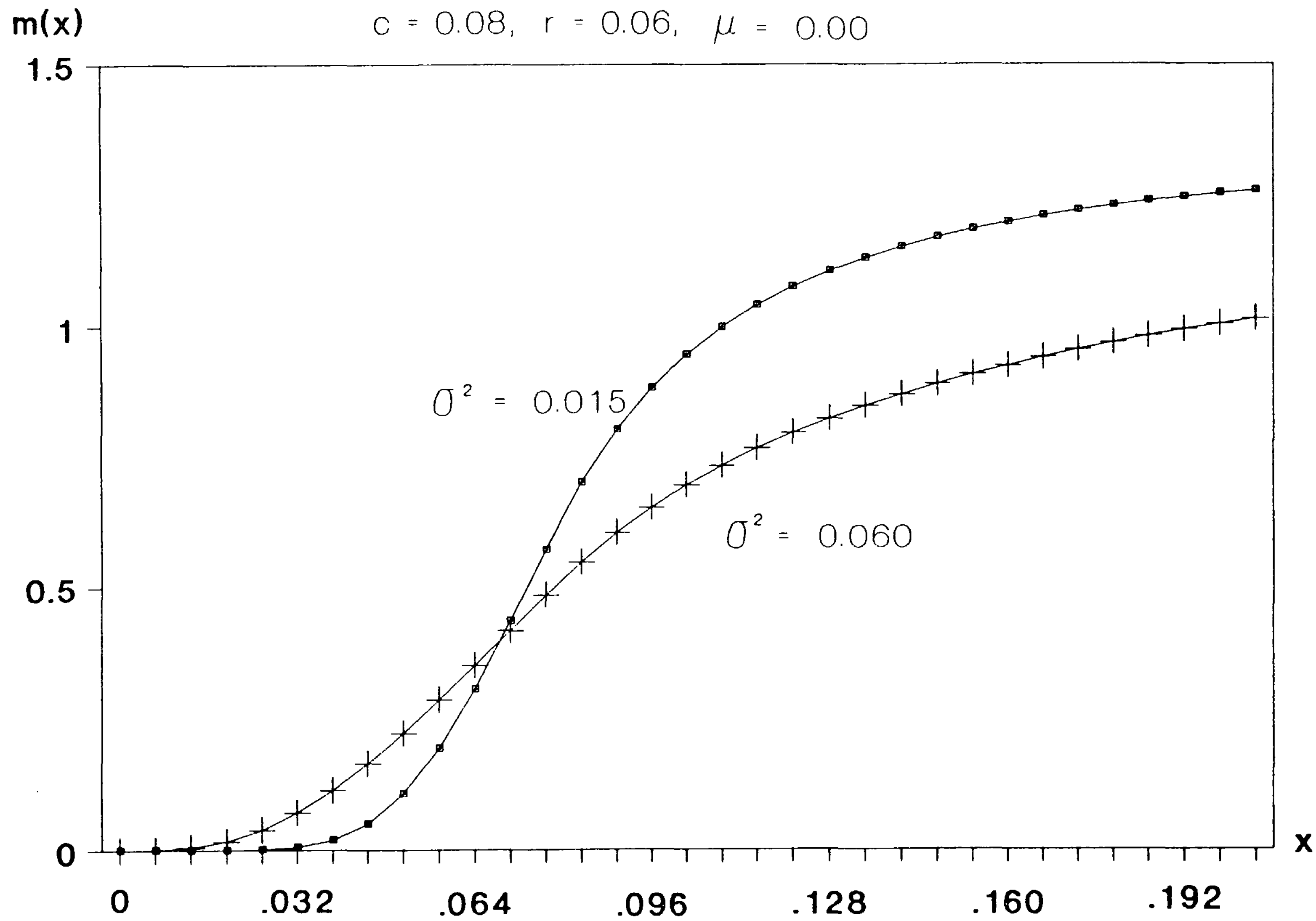
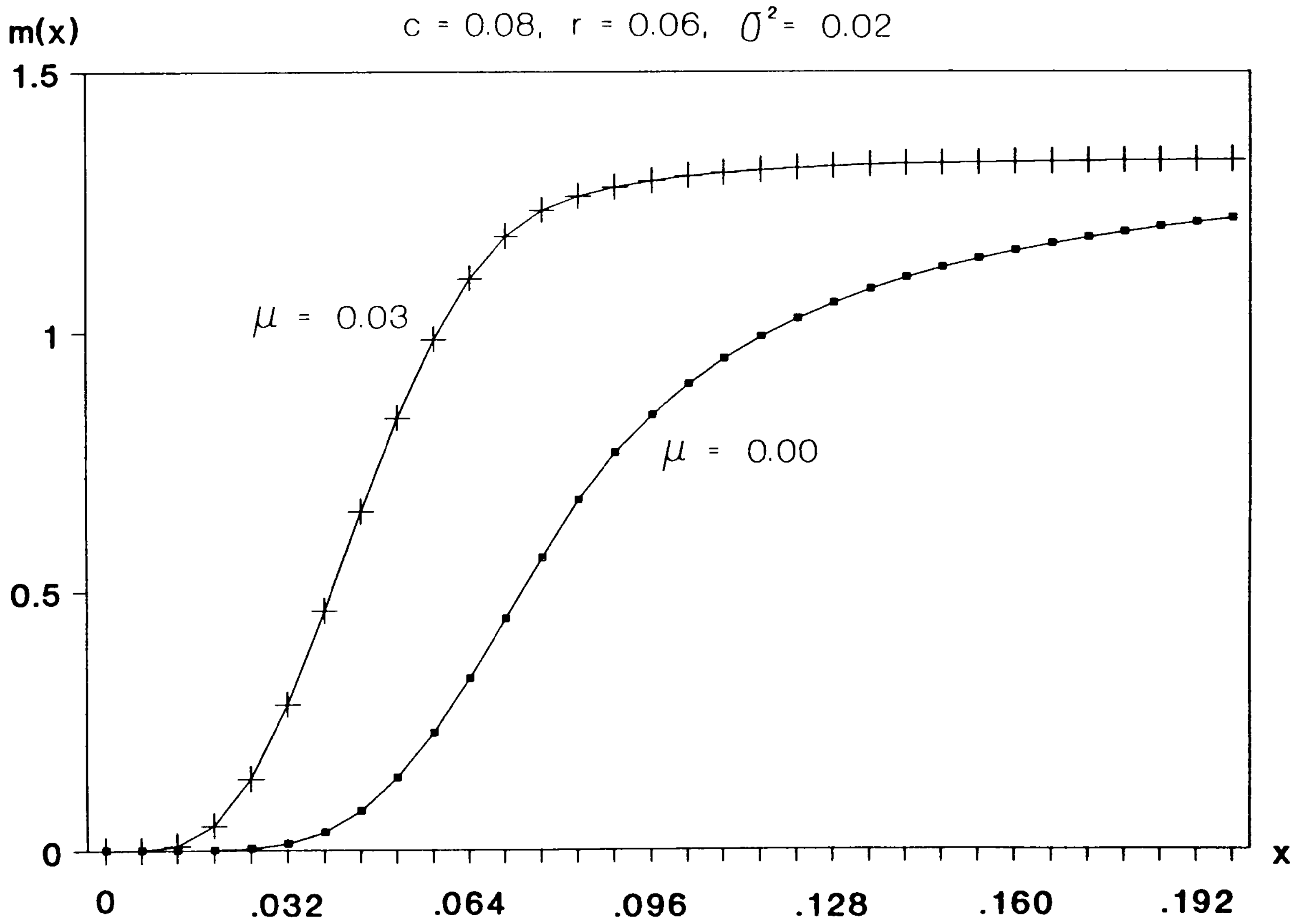

m(x)
c = 0.08, r = 0.06, μ = 0.00
1.5
1
0.5
0
σ^2 = 0.015
σ^2 = 0.060
0
.032
.064
.096
.128
.160
.192
x

Figure 5.


c = 0.08, r = 0.06, σ^2 = 0.02
m(x)
1.5
1
0.5
0
μ = 0.03
μ = 0.00
0
.032
.064
.096
.128
.160
.192
x

the relative increase in value, to be high. Different conflicts arise in the two cases.

Take the absolute value first. This rises most for those middle-ranking creditors whose claims are 'just in the money.' They will be the ones most likely to support such a policy shift by industrial countries. In contrast, very junior or very senior creditors stand to gain little from the change, and will not expend much political lobbying for the liberalization. As to relative value, that rises fastest for the most junior creditors, so the conflict mirrors seniority.

Similar conflicts of interest arise in response to other parametric shifts. Figure 3 shows the effect of restructuring the loans at a lower coupon rate. Keeping r at 6 percent, $\mu = 0$, and $\sigma^2 = 2$ percent (the values used in Figure 1), we show the comparison between coupon rates of 10 and 8 percent. The two value functions cross. There is a critical level of x such that more junior creditors are better off with the lower coupon rate, and more senior creditors are better of with the higher coupon rate. A reduction in c has two effects: each nominal dollar's claim has entitlement to a smaller flow of contractual coupon payments, but actually receives these payments for a larger subset of the stochastic outcomes. For the relatively junior creditors, the latter effect dominates.

If official creditors of indebted countries are de facto more senior than private creditors, and all creditors are concerned with the values of their claims, one should expect the IMF and the World Bank to oppose restructuring on terms more favorable to debtor countries, and the commercial banks to support such actions. To the extent that such is not the case, either the official creditors have concerns that go beyond the market value of their claims, or the private creditors expect to make up for their losses through other channels such as taxpayer-financed baleouts.

Figure 4 shows the effect of changing the volatility parameter σ. Once again there is a conflict between junior and senior creditors. The crossing point is at a value of x smaller than c. The former are better off with a larger σ, and the latter with a smaller one. If the debtor country can vary σ by changing its policies, then the junior creditors will recommend risky policies, and the senior creditors, safe ones. The intuition is simple; for junior creditors far out of the money, a riskier distribution of the debtor's prospects increases the probability of getting paid. They are not concerned about bad incomes becoming worse; they don't get paid in either of those cases.

Figure 5 shows the effect of changing the trend parameter μ. A higher μ benefits all creditors, but to different extents. Those with x a little below c gain the most in absolute terms, while the most junior ones gain the most in relative terms. If an increase in μ can be achieved at some cost, these are the creditors who will be most willing to contribute to such costs.

IV. Concluding Comments

We have considered the problem of market valuation of illiquid debt using assumptions that fit the country debt context better than do those of previous work. When we distinguish between the riskless interest rate and the coupon rate on the debt, and when we allow for rollover of arrears in repayment, the problem becomes quite complex, and textbook formulas for option pricing no longer apply. But the correct dynamic programming treatment is not hard in principle. The solutions can be characterized using functions that are well known in other uses, and are easy to compute numerically. The results are intuitive, and cast new light on various conflicts of interest among creditors.

Our calculations offer some insights into the determinants and the nature of intercreditor conflicts, and have potential implications for the process of country debt renegotiation, for burden-sharing among creditors, and for policy issues in debtor countries. Of course the results depend on the particular assumptions built.

The same technique can also be used for valuing other kinds of claims. When debt and equity are the two types, we showed how equity can be valued as a residual. This opens the way for the treatment of issues like swaps in a richer dynamic setting than was possible using other techniques of analysis.

The model needs more substantial extension to endogenize various magnitudes we specified as exogenous, most notably the coupon rate on the debt. These are topics for future research.

Technical Appendix

Here we give details of the solution procedure sketched in the text.

1. Derivation of the differential equations

Recall that the repayment capacity X satisfies

$$dX/X = \mu dt + \sigma dw, \tag{13}$$

and the debt stock D evolves according

$$dD = \begin{cases} 0 & \text{when } X \geq cD. \\ (cD-X)dt & \text{when } X < cD. \end{cases} \tag{14}$$

Then the value function V is defined by

$$V(X,D) = E\left\{\int_0^\infty \min(X_t, cD_t)\, e^{-rt}\, dt \;\middle|\; (X_0,D_0) = (X,D)\right\}. \tag{15}$$

Consider the cases $X > cD$ and $X < cD$ separately. In each case, we split the right-hand side integral as in Dynamic Programming. Separate out the contribution for an initial time interval of length dt. The integral starting at dt is expressed as a continuation value function, which is the same function V but with arguments evolved according to (13) and (14). When $X > cD$, only X changes, and we have

$$V(X,D) = cD\, dt + E\, [V(X + dX,D)\, e^{-rdt}].$$

Expanding the expectation on the right-hand side using Itô's Lemma, and simplifying,

$$0 = cDdt - rV(X,D)dt + \mu X V_X(X,D)dt + \frac{1}{2}\sigma^2 X^2 V_{XX}(X,D)dt + o(dt),$$

where o(dt) collects all terms that go to zero faster than dt. Dividing by dt and taking limits, we get the partial differential equation

$$\frac{1}{2}\sigma^2 X^2 V_{XX} + \mu X V_X - rV + cD = 0. \tag{16}$$

When $X < cD$, both X and D change, and

$$V(X,D) = cD\, dt + E[V(X + dX,D + dD)\, e^{-rdt}].$$

Expanding and simplifying similarly, we find

$$\frac{1}{2}\sigma^2 X^2 V_{XX} + \mu X V_X + (cD-X)\ V_D - rV + X = 0. \tag{17}$$

It is obvious from (13-15) that V is homogeneous of degree 1 in (X,D). Therefore, the average value V/D and the marginal value V_D are each homogeneous of degree zero in (X,D), so they can be expressed as functions of the ratio (X/D) alone. Define

$$x = X/D, \quad v(x) = V(X,D)/D, \quad m(x) = V_D(X,D). \tag{18}$$

Then

$$V_X(X,D) = v'(x), \quad V_{XX}(X,D) = v''(x)/D, \quad V_D(X,D) = v(x) - xv'(x). \tag{19}$$

Substituting these in (16-17), we get a pair of ordinary differential equations for v. When x > c, we have

$$\frac{1}{2}\sigma^2 x^2 v''(x) + \mu x v'(x) - rv(x) + c = 0. \tag{20}$$

When x < c,

$$\frac{1}{2}\sigma^2 x^2 v''(x) + (\mu-c+x)xv'(x) - (r-c+x)v(x) + x = 0. \tag{21}$$

Differentiating (16-17) with respect to D, we get a pair of partial differential equations for $V_D(X,D)$ in the two regions X > cD. Using homogeneity, they can be transformed into a pair of ordinary differential equations for m(x). When x > c,

$$\frac{1}{2}\sigma^2 x^2 m''(x) + \mu x m'(x) - rm(x) + c = 0, \tag{22}$$

and when x < c,

$$\frac{1}{2}\sigma^2 x^2 m''(x) + (\mu-c+x)xm'(x) - (r-c)m(x) = 0. \tag{23}$$

These are subject to various boundary conditions. By Theorem 4.4.9 of Karatzas and Shreve (1988), v is continuously differentiable. Therefore, at x = c its values and derivatives from the left and the right must be equal. Thus, we have the Value Matching Condition v(c-) = v(c+) and the Smooth Pasting Condition v'(c-) = v'(c+). A similar argument gives Value Matching and Smooth Pasting for m at c.

Next, when $X \ll cD$, the prospect of full payment is remote, and V(X,D) can be approximated by the expected discounted present value of X, namely $X/(r-\mu)$. Thus, as x goes to zero, so does v(x), and therefore m(x). Finally, when $X \gg cD$, full payment is almost sure to persist for a long time, and we have the approximation $V(X,D) \approx cD/r$. Therefore as x goes to ∞, v(x) and m(x) both go to c/r.

2. Solution for m(x)

Equation (22) for the region $x > c$ is simple. The general solution of the homogeneous part is

$$m(x) = B\, x^{\beta} - Ax^{-\alpha},$$

where A and B are constants to be determined, and $-\alpha$ and β are roots of the quadratic equation

$$q(\xi) \equiv \frac{1}{2}\, \sigma^2\, \xi(\xi-1) + \mu\xi - r = 0 \tag{24}$$

Note that since $q(\pm\infty) = \infty > 0$, and $q(0) = -r < 0$ and $q(1) = -(r-\mu) < 0$. Therefore, one root $(-\alpha)$ is negative and the other (β) is larger than one, and therefore positive. A simple particular solution of (A.10) is $m(x) = c/r$. The general solution is the sum

$$m(x) = B\, x^{\beta} - A\, x^{-\alpha} + c/r.$$

Since m(x) goes to c/r as $x \to \infty$, we must have $B = 0$. Thus, for $x > c$, we have

$$m(x) = c/r - A\, x^{-\alpha}, \tag{25}$$

where A remains to be determined.

The equation (22) for the range $x < c$ is harder. Make the substitution $m(x) = x^{\theta}n(x)$, where θ and n(x) are to be determined. This yields

$$x^{\theta}\, n(x)\left\{\frac{1}{2}\, \sigma^2\theta(\theta-1) + (\mu-c)\theta - (r-c)\right\}$$

$$+ x^{\theta+1}\left\{\frac{1}{2}\, \sigma^2 x n''(x) + [\sigma^2\theta+\mu-c+x]n'(x)+\theta n(x)\right\} = 0.$$

Now choose θ to get rid of the first line of this equation. For this, it should be a root of the quadratic equation

$$Q(\xi) \equiv \frac{1}{2}\, \sigma^2\, \xi(\xi-1) + (\mu-c)\xi - (r-c) = 0. \tag{26}$$

Note that

$$Q(\pm\infty) = \infty > 0, \quad Q(0) = c - r > 0, \; Q(1) = -(r-\mu) < 0.$$

Also

$$Q(\beta) = -c(\beta-1) > 0,$$

where $\beta > 1$ is the positive root of the quadratic (24). Therefore, one root of $Q(\xi) = 0$, say λ, lies between 0 and 1, while the other, say ν, is greater than β and therefore greater than 1.

When we choose $\theta = \lambda$ or ν, the equation for n(x) reduces to

$$\frac{1}{2}\sigma^2 x n''(x) + [\sigma^2\theta+\mu-c+x]n'(x) + \theta n(x) = 0.$$

Another substitution, $z = -2x/\sigma^2$, reduces it further to

$$zn''(z) + [\omega(\theta)-z]\, n'(z) - \theta n(z) = 0, \tag{26}$$

where $\omega(\theta) = 2\theta + 2(\mu-c)/\sigma^2$. This is a standard form known as Kummer's Equation, and has as its solution the confluent hypergeometric function $H(z,\theta,\omega(\theta))$; see Erdelyi et. al (1953) and Slater (1960).

Corresponding to each of the two choices of θ, we get a solution for n(z). Reversing the successive transformations, the general solution for m(x) is

$$m(x) = A'\, x^\lambda\, H(-2x/\sigma^2,\lambda,\omega(\lambda)) + B'\, x^\nu\, H(-2x/\sigma^2,v,\omega(\nu)),$$

where A' and B' are constants to be determined.

This solution must be valid for $x < c$, and consideration of its behavior as x goes to 0 helps rule out the smaller root λ. Recall that

$$v(x) - xv'(x) = m(x)$$

so

$$\frac{d}{dv}\frac{v(x)}{x} = \frac{xv'(x)-v(x)}{x^2} = -\frac{m(x)}{x^2}$$

and

$$\frac{v(x)}{x} - \frac{v(c)}{c} = \int_x^c \frac{m(x)}{x^2}\, dx.$$

Now consider the integrand for small x. As x goes to zero, $H(-2x/\sigma^2, \theta, \omega)$ goes to 1 for any θ and ω. If $A' \neq 0$, then m(x) behaves like x^λ. Therefore, the integral behaves like

$$-A' \; x^{(\lambda-1)}/(\lambda-1) = A' \; x^{-(1-\lambda)}/(1-\lambda).$$

Since $0 < \lambda < 1$, this goes to ∞ if $A' > 0$ and to $-\infty$ if $A' < 0$. Both of these outcomes are impermissible, the former because

$$V(X,D) = E \int_0^\infty \min(X_t, cD_t) \; e^{-rt} \, dt$$

$$\leq E \int_0^\infty X_t \; e^{-rt} \, dt = X/(r-\mu),$$

so $v(x)/x \leq 1/(r-\mu)$ is bounded above.

Thus we rule out the term corresponding to λ, and write the solution as

$$m(x) = B \; x^v \; H(-2x/\sigma^2, v, w), \tag{28}$$

where we have simplified the notation, writing B instead of B' and ω instead of $\omega(\nu)$.

Finally, the Value Matching and Smooth Pasting conditions determine the constants A and B. The Value Matching condition is

$$v(c+) = c/r - A \; c^{-\alpha} = B \; c^\nu \; H(-2c/\sigma^2, \nu, \omega) = v(c-),$$

and Smooth Pasting condition is

$$v'(c+) = \alpha A c^{-\alpha-1} = B[\nu c^{\nu-1} \; H(-2c/\sigma^2, \nu, \omega) - (2/\sigma^2) c^\nu \; H'(-2c/\sigma^2, \nu, \omega)] = v'(c-).$$

Using a standard property of the confluent hypergeometric function (easily verified by differentiating the series in equation (10) in the text term by term), we can write the Smooth Pasting Condition as

$$\alpha A c^{-\alpha-1} = B_c{}^\nu [(\nu/c) \; H(-2c/\sigma^2, \nu, \omega) - (2\upsilon/\omega\sigma^2) \; H(-2c/\sigma^2, \nu+1, \omega+1)].$$

These linear equations are easy to solve for A and B. Once m(x) is known, v(x) can be found by integrating as in (27).

3. Solution for v(x)

As with m(x), the solution for v(x) is simple in the region $x > c$. We have

$$v(x) = c/r - A \; x^{-\alpha}, \tag{29}$$

where A is a constant to be determined (not the same as the A in the solution for m(x)).

In the region $x < c$, we first solve the homogeneous part of (21), namely

$$\frac{1}{2}\sigma^2 x^2 v''(x) + (\mu-c+x)xv'(x) - (r-c+x)v(x) = 0.$$

The procedure is similar to that used above for m(x). Make the substitution $v(x) = x^\theta w(x)$, where θ and w(x) are to be determined. Choose θ to be a root of the quadratic $Q(\xi) = 0$. Define $w(\theta)$ as before. Finally, make the substitution $z = -2x/\sigma^2$. This gives

$$zw''(z) + [\omega(\theta)-z]\, w'(z) - (\theta-1)\, w(z) = 0. \tag{30}$$

Once again this is Kummer's Equation, and has the confluent hypergeometric solution $H(z,\theta-1,\omega(\theta))$. Then the general solution of the homogeneous part of the equation for v(x) (sometimes called the complementary function) is

$$v_c(x) = A'\, x^\lambda\, H(-2x/\sigma^2,\lambda-1,\omega(\lambda)) + B'\, x^\nu\, H(-2x/\sigma^2,\, \nu-1,\omega(\nu)). \tag{31}$$

The constants A', B' are to be determined (again not the same as the A', B' of m(x).)

A particular solution to the non-homogeneous equation (21) can be found using the method of variation of parameters. Write (21) as

$$a_0(x)v''(x) + a_1(x)v(x) - a_2(x)v(x) - b(x) = 0, \tag{32}$$

and denote the two independent solutions of the homogeneous part by

$$v_1(x) = x^\lambda\, H(-2x/\sigma^2,\, \lambda-1,\omega(\lambda)),\quad v_2(x) = x^\nu\, H(-2x/\sigma^2,\nu-1,\omega(\nu)). \tag{33}$$

Now look for a particular solution of (32) of the form

$$v_p(x) = f_1(x)v_1(x) + f_2(x)v_2(x), \tag{34}$$

where $f_1(x)$ and $f_2(x)$ are undetermined <u>functions</u>. The requirement that $v_p(x)$ should satisfy (32) imposes one condition on $f_1(x)$ and $f_2(x)$, but leaves one degree of freedom. We can use it to make the resulting solution as simple as possible.

Differentiating (34) and substituting into (32), we find

$$v_1'(x)\, f_1'(x) + v_2'(x)\, f_2'(x) = b(x)/a_0(x).$$

We impose the additional condition for simplification:

$$v_1(x)\ f_1'(x) + v_2(x)\ f_2'(x) = 0.$$

Solving these,

$$f_1'(x) = -\frac{b(x)v_2(x)}{a_0(x)W(x)},\quad f_2'(x) = \frac{b(x)v_1(x)}{a_0(x)W(x)}, \tag{35}$$

where $W(x)$ is the Wronskian of $v_1(x)$ and $v_2(x)$:

$$W(x) = v_1(x)\ v_2'(x) - v_2(x)\ v_1'(x),$$

and is non-zero except at $x = 0$ by the independence of v_1 and v_2.

Now (35) can be solved by simple quadratures. It proves convenient to choose x and c as the limits of integration, and we get the particular integral

$$v_p(x) = v_1(x) \int_x^c \frac{b(x)v_2(x)}{a_0(x)W(x)}\,dx - v_2(x) \int_x^c \frac{b(x)v_1(x)}{a_0(x)W(x)}\,dx. \tag{36}$$

The advantage of this choice is that $v_p(c)$ and $v_p'(c)$ are both zero, which allows a partial closed form solution for the Value Matching and Smooth Pasting conditions.

Thus the Value Matching Condition becomes

$$A'\ c^{\lambda}H(-2c/\sigma^2,\lambda-1,\omega(\lambda)) + B'c^{\nu}H(-2c/\sigma^2,\nu-1,\omega(\nu)) = c/r - Ac^{-\alpha}, \tag{37}$$

while the Smooth Pasting condition is

$$\alpha Ac^{-\alpha-1} = A'[\lambda c^{\lambda-1}H(-2c/\sigma^2,\lambda-1,\omega(\lambda)) - (2/\sigma^2)c^{\lambda}H'(-2c/\sigma^2,\lambda-1,\omega(\lambda))]$$
$$+ B'[\nu c^{\nu-1}H(-2c/\sigma^2,\nu-1,\omega(\nu)) - (2/\sigma^2)c^{\nu}H'(-2c/\sigma^2,\nu-1,\omega(\nu))]. \tag{38}$$

These are two equations in three unknowns, A, A', and B', leaving one degree of freedom. As with m(x), this is fixed using the boundedness of v(x)/x as x goes to 0. There the condition simply got rid of the term involving the smaller root λ. Here the argument is more complicated, and so is the result.

As x tends to 0, the integrand in $f_1(x)$ behaves like $x^{-\lambda}$ and that in $f_2(x)$ behaves like $x^{-\nu}$. Therefore

$$v_p(x) \sim x^{\lambda} \int x^{-\lambda}\, dx - x^{\nu} \int x^{-\nu}\, dx$$

$$= x^{\lambda} \left\{ \frac{x^{1-\lambda}}{1-\lambda} + c_1 \right\} + x^{\nu} \left\{ \frac{x^{1-\nu}}{1-\nu} + c_2 \right\}$$

$$= c_1 x^{\lambda} - c_2 x^{\nu} - x(\nu-\lambda)/[(1-\lambda)(1-\nu)],$$

where c_1 and c_2 are constant of integration. Since $0 < \lambda < 1 < v$, the leading term in this $c_1 x^{\lambda}$. Then $v_p(x)/x$ goes to ∞. Turning to the complementary function in (30), write

$$v_c(x)/x = A'v_1(x)/x + B'v_2(x)/x$$

and note that $v_2(x)/x$ goes to 0 as x goes to 0. If

$$v(x)/x = v_c(x)/x + v_p(x)/x = A'v_1(x)/x + B'v_2(x)/x + v_p(x)/x$$

is to stay bounded, then we must have

$$(A'v_1(x)+v_p(x))/x$$

bounded. For this we need

$$A' = - \lim_{x \to 0} v_p(x)/v_1(x). \tag{39}$$

Since $v_p(x)$ and $v_1(x)$ are in an explicit form that can be evaluated numerically, we can find A', and then solve (36-37) for B' and A, thus completing the solution for v(x). Then m(x) is easy to derive using (12).

Our numerical calculations were done independently for m(x) using the method of Section 3 and for v(x) using the method of Section 4. The results matched to within 1 percent for all parameter values, and much more closely for most.

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