0440

IMF WORKING PAPER

WP/91/126

INTERNATIONAL MONETARY FUND

European Department

Government Ponzi Games and Debt Dynamics Under Uncertainty

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December 1991

Abstract

We investigate the conditions for sustainability of debt roll-over schemes under uncertainty. In contrast with the requirements identified in recent research, we show that a necessary and sufficient condition for sustainability of such schemes is that the asymptotic interest rate on government debt be lower than the asymptotic growth rate of the economy, a natural extension of a familiar criterion in a deterministic framework. However, we also show that for realistic parameter values, Ponzi games that are sustainable in the long run may display explosive patterns over relatively long horizons. This may explain why governments may be reluctant to play Ponzi games even when they are feasible in the long run.

JEL Classification Numbers:
E60, H62

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* We thank Giampaolo Galli and Erich Spitäller for comments. All remaining errors are our own.

Contents

		Page
Summary		iii
I.	Introduction	1
II.	Solvency Conditions in a Nonstochastic Environment	2
III.	Debt Ponzi Games Under Uncertainty	6
IV.	Debt Stabilization in Finite Horizons	11
V.	Numerical Simulations	13
VI.	Concluding Remarks	16
References		17
Charts		
	1. Convergence Band of the Debt Ratio	12a
	2. Convergence Under the Initial Value of the Debt Ratio	12a
	3. Probability of Exceeding a Given Ceiling	16a
Table		
	1. Correlation of GDP Growth Rates	14

Summary

This paper reconsiders the conditions under which a government may engage in debt rollover schemes by financing interest payments through the issue of new debt. Output growth rates in excess of interest rates on government debt have traditionally been considered grounds for sustaining such schemes. A government may avoid debt repayment, or even run primary deficits forever, and yet may maintain a bounded debt-to-income ratio. Recent research pointing at the stronger constraints placed on government behavior by uncertainty has challenged this possibility. This paper shows that uncertainty does not reduce the scope for debt rollover when the appropriate criterion for solvency is used, namely, the condition that the debt-to-income ratio converges almost surely in the long run. In this case, the government is solvent when the average asymptotic growth rate of the economy exceeds the average asymptotic interest rate on debt, a natural extension of a familiar criterion in a deterministic environment.

Convergence to the long-run outcome may be, however, a slow process. For realistic parameter values, fiscal plans that are stable in the long run may give rise to short-run paths of the debt ratio that resemble those of unsustainable rules. This circumstance explains the difficulty that governments may encounter in running debt rollover schemes, if a financial crisis is triggered when the debt ratio exceeds a finite bound. It also explains the observed poor performance of debt ratios as indicators of fiscal sustainability.

I. Introduction

Can governments run primary deficits forever and get away with it? The standard answer is that it all depends on the relation between the interest rate on public debt and the growth rate of output. When the former exceeds the latter, then permanent deficits 1/ imply that debt dynamics are unstable: sooner or later the government will have to resort to increased taxation, expenditure cuts or monetary financing. On the contrary, if the output growth rate exceeds the interest rate, then permanent deficits are sustainable, for the debt to output ratio converges to a finite limit, and "taxes need never be levied and may indeed be negative forever" (Buiter (1985), page 33). In this case the government would face no binding solvency constraint, and would be able to service its debt simply by issuing new debt (a 'Ponzi game'), a somewhat puzzling state of affairs.

Of course, it could be argued that the interest rate is not independent of the stance of fiscal policy. Even if, at a given moment, the interest rate falls short of the economy's growth rate, the pressure of increasing debt may eventually reverse the situation. This argument has found theoretical support (see Miller and Sargent (1984), Masson (1985), Weil (1987)) but, in the absence of adequate empirical evidence on the effect of public debt on interest rates, it has not been considered conclusive. A second objection moved against the possibility of a growth rate in excess of the interest rate has been based on its inconsistency with dynamic efficiency. 2/ Even this argument, however, is not sufficient to rule out government Ponzi games: in the presence of output uncertainty, efficiency is compatible with a positive output growth -interest rate differential (see, for instance, Blanchard and Weil (1991)).

A third argument against the feasibility of Ponzi-like games has been raised recently by Blanchard and Weil (1991) and Galli and Giavazzi (1991): under conditions of output uncertainty, a positive differential between output growth and interest rate on debt is not a sufficient condition for the asymptotic boundedness of the *expected* debt to output ratio. If this explodes asymptotically, then the debt Ponzi game cannot be sustained. Only when the interest rate is sufficiently smaller than the output growth rate the expected debt to income ratio would converge to a finite value. The resulting restrictions on government behavior under uncertainty would therefore be more stringent than those prevailing under certainty.

In this paper we focus on this last argument. We reconsider the conditions for sustainability of budget deficits and how these conditions are modified by the introduction of uncertainty. In Section II we review

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1/ Unless otherwise stated, throughout the paper the term "deficit" refers to the "primary deficit".

2/ In this respect, recent empirical studies (particularly Abel et al. (1989)) tend to dismiss earlier evidence on the existence of dynamic inefficiency, at least for major industrial countries.

the deterministic case and the requirements for debt sustainability under two alternative definitions of solvency: the No-Ponzi-Game condition and the boundedness of the debt to income ratio condition. These two conditions are equivalent, in steady state, only when the interest rate exceeds the growth rate, but the latter is less stringent when the opposite is true. To the extent that the appropriate yardstick to evaluate government solvency is the boundedness of the debt ratio, the puzzle of permanent sustainable deficits arises.

In Section III we discuss the application of the boundedness condition in a stochastic environment. In contrast with Blanchard and Weil (1991) and Galli and Giavazzi (1991), we argue that the appropriate criterion for government solvency under uncertainty, given a certain policy rule, is whether the debt to output ratio is on a stable path, and not whether the limit of the cross-path expectation is bounded. Following this approach, we show that the boundedness condition in a stochastic environment represents a natural extension of the same condition in a nonstochastic environment, namely that the asymptotic average growth rate exceeds the asymptotic average interest rate on government debt. If this condition prevails, then the debt to output ratio converges with probability one, for quite general specifications of the stochastic process driving the economy and of the model determining the interest rate.

In Sections IV and V we focus on the dynamics of the debt ratio in finite horizons and we show that before eventual convergence the probability that the ratio hits values substantially higher than its initial level remains high for realistic parameter values. This fact may explain why, in practice, governments may have trouble in playing Ponzi-like games. It also implies that observed trends of the debt to income ratio may be poor indicators of government solvency, a finding which is consistent with the results of recent empirical studies. 1/ Section V concludes by summarizing the main findings of the paper.

II. Solvency Conditions in a Nonstochastic Environment

A necessary condition for sustainability of a certain policy rule (e.g. a permanent deficit) is that, under that rule, the government is solvent. How do we define solvency? Consider first the effect of imposing on the government the No-Ponzi-Game (NPG) condition, the condition traditionally applied for solvency of private agents. According to this condition, an

1/ See for instance Corsetti and Roubini (1991) and Horne (1991).

agent is solvent if the present value of its long-run liability is nonpositive. This condition can be written as

$$(1) \quad \lim_{T\to\infty} D_T/(1+r)^T \leq 0$$

where D_T is the level of debt at the beginning of period T and r is the interest rate. Equivalently, by expressing debt as a fraction of income, the NPG condition can be written as

$$(2) \quad \lim_{T\to\infty} d_T[(1+g)/(1+r)]^T \leq 0$$

where $d_T = D_T/Y_T$, Y_T is output, and g is its constant growth rate.

In the absence of monetary financing of the deficit, 1/ the dynamic government budget identity can be written as

$$(3) \quad D_{t+1} = [D_t - S_t](1+r)$$

where S_t is the primary surplus at time t. This can be rewritten as

$$(4) \quad D_t = S_t + D_{t+1}/(1+r)$$

Expression (4) can be solved recursively forward, from time zero. This yields

$$(5) \quad D_0 = \lim_{T\to\infty} \left[\sum_{t=0}^{T} \frac{S_t}{(1+r)^t} + \frac{D_T}{(1+r)^T} \right]$$

Combining (5) and (1), the NPG condition can be written as

$$(6) \quad D_0 - \lim_{T\to\infty} \left[\sum_{t=0}^{T} \frac{S_t}{(1+r)^t} \right] \leq 0$$

This condition requires the present value of the planned surplus flow to be at least as large as the current level of debt. The same condition can be stated by expressing debt and surplus as a fraction of output. The NPG condition then becomes

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1/ We assume the absence of monetary financing in order to focus on debt roll-over paths along which the government avoids the "painful trade-off" (Fischer and Easterly (1990)) between raising taxes and increasing inflation. We also do not consider additional complications, such as capital gains or losses and other factors analyzed, for example, by Buiter (1985). The treatment in this section can be given in continuous time, but we have preferred the discrete time approach for consistency with the simulations in later sections.

$$(7) \quad d_0 - \lim_{T\to\infty} \sum_{t=0}^{T} s_t[(1+g)/(1+r)]^t] \leq 0$$

where $s_t = S_t/Y_t$. How does the sign of the differential (r-g) influence the sustainability of a fiscal plan along a steady state? Along a steady-state the surplus will be a constant fraction of income, that is $s_t = s$. It is also easy to verify that the steady-state dynamics of the debt ratio are characterized by a saddle path $d = s(1+r)/(r-g)$. The saddle path is stable when $g>r$, and unstable when $r>g$.

When $r>g$, the series $\Sigma_T[(1+g)/(1+r)]^T$ converges to $(1+r)/(r-g)$, so that (7) requires

$$(8) \quad d_0 \frac{(r-g)}{(1+r)} \leq s$$

According to condition (8), the government is solvent if the steady-state primary surplus is sufficiently large with respect to the initial level of debt. Since $r>g$, then $s>0$: an initial issue of debt, followed by a policy of permanent (or zero) deficits, is not sustainable in steady state. Regarded as a constraint on the initial value of debt, for a given level of the steady-state primary surplus, condition (8) requires the initial debt ratio to be smaller than its saddle-point $s(1+r)/(r-g)$.

Consider now the case in which $r<g$. Since the series $\Sigma_T[(1+g)/(1+r)]^T$ diverges, the left-hand side of (7) tends to minus infinity for any positive value of s, so that the solvency condition is always met for any (finite) initial level of debt. The opposite is true if $s<0$: the constraint is never met, and the government is insolvent. Thus, even when $r<g$, a stream of permanent deficits is not sustainable on a steady state if the NPG condition is used as the relevant criterion for government solvency.

In order to admit that the government can engage in a policy of permanent deficits, following an initial issue of debt, we need to adopt a milder definition of solvency, namely that the debt ratio remains bounded as the horizon goes to infinity. This is the condition

$$(9) \quad \lim_{T\to\infty} D_T/Y_T \equiv \lim_{T\to\infty} d_T \leq k \qquad \text{for some } k>0 \quad \underline{1}/$$

When $r>g$, condition (9) is formally stronger than the NPG condition (2), which only requires the debt ratio to grow at a rate smaller than $(1+r)/(1+g)$, possibly forever. As we have seen, however, the only steady-state paths which satisfy the NPG condition (8) are those starting from a level of the debt ratio lower than its unstable saddle-point, i.e. from

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1/ For future reference, observe that in a deterministic environment it is irrelevant whether the requirement be that d_T be bounded above by k or that $1/d_T$ be bounded below by $1/k$.

levels at which the boundedness condition is also satisfied. 1/ When $r>g$, therefore, the NPG condition is weaker than the condition requiring the debt ratio to remain bounded, only to the extent that it allows a broader range of short-run fluctuations. The two conditions, however, are equivalent in terms of long-run (or steady-state) sustainability. When $r<g$, however, the boundedness condition (9) is weaker than the NPG condition, even along a steady state: given the stability of the saddle path when $r<g$, it only requires that $D_t/D_{t-1}<1+g$, against the stronger condition that $D_t/D_{t-1}<1+r$.

To appreciate the extent to which the boundedness condition relaxes the requirements on the fiscal budget, notice that (9) imposes the following constraint on the steady-state primary balance:

$$(10) \quad s \geq k(r-g)/(1+r)$$

According to condition (10), when $g>r$, a budget deficit smaller than $k(g-r)/(1+r)$ can be sustained indefinitely, while yielding eventual convergence to a steady state value of the debt to income ratio smaller than k. In fact, if it is required only that the long-run level of the debt ratio be bounded away from infinity (i.e., $k<\infty$), then any primary deficit can be indefinitely sustained when the economy's growth rate exceeds the interest rate on debt.

The use of the boundedness condition as a criterion for government solvency is the essential reason for this somewhat uncomfortable result. The reason why such condition has been traditionally adopted, is the role that the economy's output plays as a collateral for government debt (see, for instance, Buiter (1983)). If the government has, at least in principle, the right to extract a fraction (even arbitrarily small) of national output in the form of net taxes, then the present value of this right is infinite whenever $g>r$, and will always exceed the value of its liability. 2/ Thus, contrary to the NPG condition, the boundedness criterion has the financially appealing feature of not ruling out fiscal plans along which the value of government's collateral always outpaces its liabilities.

How is the condition that the debt to income ratio remain bounded to be interpreted under uncertainty? The problem is to establish whether government's liabilities, given a stochastic process driving the economy and a given fiscal plan, can be expected to outstrip its collateral in the long run. Accordingly, it has been suggested that the appropriate way to

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1/ From a different stand-point, for $d_0>s(1+r)/(r-g)$, the asymptotic growth rate of d is exactly $(1+r)/(1+g)$, which violates the NPG requirement that the growth rate of d be *always* smaller than this value.

2/ This is equivalent to saying that the value of the taxation authority of the Government should not be evaluated as the discounted stream of *planned* primary balances (which would be negative for a policy of permanent deficits), but with respect to the *potential* primary balances (i.e., net taxes) that the Government could extract from the nation.

collapse the uncertainty about all the possible paths which the economy can follow is to consider the limit of the cross-path expected value of the debt to income ratio. The asymptotic *cross-path* expectation, however, need bear little information on the expected eventual outcome from a *single path*. Indeed, the next section shows that the criterion of convergence of the expected value of the debt ratio, used in Blanchard and Weil (1991) and Galli and Giavazzi (1991) among others, rules out fiscal plans in which the debt to income ratio converges with probability one.

III. Debt Ponzi Games Under Uncertainty

As an illustration of the implication of uncertainty for the issue of debt stabilization, consider the following example from Galli and Giavazzi (1991). A Lucas-type economy is subject to output uncertainty. The output process, without loss of generality, can be written in multiplicative form:

(11) $Y_t/Y_{t-1} = (1+g)v_t$ 1/

Let output shocks be lognormally distributed, that is

(12) $\ln v_t \sim N(0,\sigma^2)$

and let utility be logarithmic. Asset returns can be obtained in standard fashion from the Euler equation. For a riskless asset (e.g. public bonds), the rate of return (1+r) can be solved as

(13) $$(1+r) = \frac{(1+g)}{\beta e^{\sigma^2/2}}$$

where β is the representative consumer's discount factor, with $0<\beta<1$. Notice that for sufficiently high uncertainty, that is for $\sigma^2>-2\ln\beta$, the economy's (average) growth rate g exceeds the rate of interest on debt. Since consumption and asset returns are chosen optimally by an infinitely lived representative agent, the economy is dynamically efficient.

Suppose now that the government runs a Ponzi game, where it services its debt simply by issuing new bonds and never running a primary surplus. 2/ Government debt then evolves according to

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1/ In general, $Y_t/Y_{t-1}=(1+\tilde{g}_t)=(1+g)[(1+\tilde{g}_t)/(1+g)]\equiv(1+g)v_t$, where $v_t\equiv(1+\tilde{g}_t)/(1+g)$. If v_t has a stationary distribution, it is also convenient to normalize $E(\ln v_t)=0$ by choosing $\ln(1+g)\equiv E[\ln(Y_t/Y_{t-1})]$.

2/ In what follows, we shall focus on the simple case of a 'pure Ponzi game', a roll-over scheme in which debt is issued to finance a once-for-all primary deficit, as well as to cover the subsequent interest charges. Without loss of generality, this covers the case of arbitrarily long streaks of primary deficits, which can be regarded as a sequence of overlapping Ponzi games, as well as the case of increasing primary deficits.

(14) $D_t = (1+r)D_{t-1}$

Using equation (13), and recalling that $E(1/v) = e^{\sigma^2/2}$ when the output shocks are log-normally distributed, the dynamics of the expected debt to income ratio can be expressed by the recursive equation

(15) $$E_{t-1}[d_t] = E_{t-1}[(1+r)D_{t-1}/(1+g)v_tY_{t-1}] = \\ = E_{t-1}[(1+r)/(1+g)v_t]d_{t-1} = \beta^{-1}d_{t-1}$$

By iterating the expectation in (15), one obtains $E_0(d_T)=\beta^{-T}d_0$, which diverges to infinity with T. Thus, the expected debt to income ratio explodes when $T\rightarrow\infty$, for all values of the other parameters. In particular, when uncertainty is sufficiently strong, it explodes for values of the growth rate in excess of the interest rate. If divergence of the expected debt to income ratio is indication of the unsustainability of the adopted fiscal rule, then the introduction of uncertainty seems to restrict the government's ability to run permanently Ponzi games.

The scope for tightening the requirements on fiscal policy as effect of uncertainty goes beyond the simple case of a representative agent economy. Blanchard and Weil (1991) have considered the asymptotic properties of the expected value of the debt ratio in an economy with overlapping generations and unit elasticity of intertemporal substitution. They show that when utility is logarithmic, the expected debt to GDP ratio explodes irrespective of the value of the riskless rate. In the more general case of a utility function with constant relative risk aversion, the expected value may go to zero only if agents' degree of risk aversion is sufficiently high to induce a large enough wedge between the economy's growth rate and the interest rate. The divergence of the limiting value of the expected debt to income ratio is taken by Blanchard and Weil as sufficient to rule out the feasibility of a Ponzi game.

The notion that debt boundedness be characterized by the asymptotic behavior of $E_0(d_T)$ is very natural, but produces uncomfortable implications. First of all, it seems to deprive the analysis of debt stabilization of the intuitive comparison between the interest rate on debt and the economy's growth rate. Second, as an obvious offspring of Jensen's inequality, the conditions for convergence of $E_0(d_T)$ and for divergence of $E_0(1/d_T)$ do not coincide: there are ranges of interest rates and growth rates for which both the expected value of d_T and the expected value of $1/d_T$ diverge, a situation of difficult economic interpretation. Finally, and most disturbing, simulations of model (11)-(14) indicate that the explosiveness of the expected debt ratio when the economy's average growth rate exceeds the interest rate on debt is driven by a small and decreasing number of paths, along which the ratio takes large and increasing values. Indeed, the proportion of divergent paths goes to zero as the horizon goes to infinity.

The intuition from the simulation underlies a stronger analytical result. The reason for the above mentioned counterintuitive results is the use of an inappropriate criterion for debt sustainability--the condition on convergence of $E_t(d_T)$--rather than the economically more significant criterion of almost sure (or 'strong') convergence of d_T.

The following result, which is the basic ingredient of our analysis, shows that when the growth rate of the economy exceeds the interest rate on debt (both defined as long-run, or asymptotic, averages), then the debt to income ratio converges to zero almost surely along a Ponzi scheme. Under these conditions the government can just sit on a path and wait long enough for convergence to be achieved. Rational agents, aware of the conditions for eventual convergence of the debt ratio, will thus subscribe to the 'honest' Ponzi game whenever the economy's asymptotic growth rate exceeds the interest rate on debt.

Proposition 1: If the productivity shocks are independent and identically distributed with finite expectation, the steady-state debt to income ratio converges almost surely to zero along a debt roll-over path if and only if the interest rate on debt is smaller than the asymptotic average growth rate of the economy.

Proof: From equations (11) and (14) the debt to income ratio can be written as:

$$d_T \equiv \frac{D_T}{Y_T} = \frac{(1+r)D_{T-1}}{(1+g)v_T Y_{T-1}} = [\frac{1+r}{1+g}]^2 \frac{D_{T-2}}{v_T v_{T-1} Y_{T-2}} =$$

$$= [\frac{1+r}{1+g}]^T \frac{1}{\Pi_{t=1,T} v_t} \frac{D_0}{Y_0} = d_0 [\frac{1+r}{1+g}]^T \Pi_{t=1}^T v_t^{-1}$$

$$= d_0 [\frac{1+r}{1+g}]^T e^{\ln \Pi_{t=1}^T v_t^{-1}} = d_0 e^{T ln(\frac{1+r}{1+g}) + \Sigma_{t=1}^T \ln v_t^{-1}}$$

$$= d_0 e^{T[\ln(\frac{1+r}{1+g}) - \frac{\Sigma_{t=1}^T z_t}{T}]} \qquad \text{where } z_t \equiv \ln v_t$$

Therefore, $\lim_{T\to\infty} d_T = 0$ iff $\lim_{T\to\infty} T\{\ln[(1+r)/(1+g)] - \Sigma_t z_t/T\} = -\infty$. Since the shocks v_t are i.i.d., so are their transforms z_t. By Kolmogorov Strong Law of Large Numbers (SLLN, see for instance Rao (1973), p.115), $\Sigma_t z_t/T$ converges to $E(z)$ almost surely as $T\to\infty$. Thus, $\lim_{T\to\infty} d_T = 0$ almost surely if and only if $\ln[(1+r)/(1+g)] - E(z) < 0$. This is equivalent to

$$1+r < (1+g)e^{E(z)}$$

or

$$1+r < (1+g)e^{E(\ln v)}$$

In the last expression, $(1+g)e^{E(\ln v)}$ equals one plus the asymptotic average growth rate of income along the steady-state path. This is because

$$\left(\frac{Y_T}{Y_0}\right)^{1/T} = \left[\frac{Y_0(1+g)^T \Pi_{t=1}^T v_t}{Y_0}\right]^{1/T} = (1+g)e^{\frac{\ln \Pi_{t=1}^T v_t}{T}}$$

$$= (1+g)\ e^{\sum_{t=1}^{T} \ln v_t/T}$$

However, by the SLLN, as $T\to\infty$ the power in the exponential term converges to $E(\ln v)$ almost surely. □

While Proposition 1 is proven under the assumption of a fixed interest rate on debt, this restriction is in no way necessary. Indeed, the interest rate can follow as general a stochastic process as the output shocks. Similar to the notation used for the output growth rate, write the stochastic process driving the interest rate as

(16) $(1+r_t) = (1+r)w_t$

Then, following the same procedure applied above,

(17) $\lim_{T\to\infty} d_T = \lim_{T\to\infty} d_0 e^{T \ln[\frac{1+r}{1+g} + \frac{\Sigma_t \ln w_t}{T} - \frac{\Sigma_t \ln v_t}{T}]}$

which converges to zero almost surely if and only if:

(18) $(1+r)e^{E(\ln w)} < (1+g)e^{E(\ln v)}$

As in the proof of Proposition 1, the left-hand-side of (18) is equal to (one plus) the average asymptotic interest rate on debt. 1/

Since the essential ingredient of Proposition 1 is the Strong Law of Large Numbers (SLLN), it is clear that the assumption of independent and identically distributed shocks can be weakened by correspondingly

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1/ The condition for steady-state debt convergence in Proposition 1 can be made more explicit if the distribution of the stochastic shocks and the model determining the riskless rate are known. In the simple logarithmic model discussed earlier, for instance, the asymptotic average growth rate is g, while the interest rate of debt is fixed at r, so that the condition for convergence of the debt to income ratio is simply $r<g$.

strengthening the requirements on the moments of the distribution of the productivity shocks. For instance, the Strong Law of Large Numbers holds, under the so-called "mixing conditions" (White (1984), pp. 43-51), for dependent and heterogeneously distributed random variables. So does, therefore, Proposition 1. This is particularly important when one considers models more general than the simple exchange economy illustrated above, such as models of economies with changing investment opportunities. In this case the interest rate on debt need not be a constant, nor follow a simple stochastic process.

It should also be obvious that the conditions given in Proposition 1 for the convergence (or boundedness from above) of D/Y are the same which characterize divergence (or boundedness from below) of Y/D. This is in contrast with the case in which convergence is defined in terms of existence of $\lim_{T\to\infty} E_0 d_T$. Finally, notice that since the probability mass is clustered at one single point (either zero or infinity), it follows that $E_0(\lim_{T\to\infty} d_T)$ is zero or infinity according to whether condition (18) holds or not.

The two convergence criteria discussed above can also be compared, and the extent of the difference appraised. In the general case of stochastic growth rate and interest rate (not necessarily uncorrelated), we have

(19) $E_{t-1}(d_t)=[(1+r)/(1+g)]E(w_t/v_t)d_{t-1}$

Thus, convergence of the expected value requires that

(20) $[(1+r)/(1+g)]E(w_t/v_t) < 1$

From equation (18), on the other hand, almost sure convergence requires

(21) $\dfrac{(1+r)}{(1+g)} e^{E[\ln(w_t/v_t)]} < 1$

Therefore, the conditions for convergence of the expected value of the debt to income ratio are more restrictive than those for its almost sure convergence when

(22) $\ln(E[w/v]) > E[\ln(w/v)]$

Since the logarithm is a concave function, (22) is always satisfied as a strict inequality as long as there is uncertainty about either the growth rate or the interest rate. It is thus apparent that tighter constraints for debt stabilization under uncertainty arise from imposing convergence of the expected debt to income ratio, merely as an implication of Jensen's inequality.

IV. Debt Stabilization in Finite Horizons

In Section III the solvency of the government was assessed purely in terms of the asymptotic properties of the debt ratio. In practice, it may be important to appraise the extent to which the long-run tendency imposes itself over shorter horizons. For this reason, in this section we explore the problem of the speed of asymptotic convergence, an issue which can be addressed independently of specific assumptions on the output process or the nature of the optimization problem. A calibration over realistic parameter values, considered in the next section, suggests that long-run-stable fiscal rules may display short-run dynamics which resemble quite closely those of explosive rules. This casts considerable doubt on empirical studies aimed at extrapolating the sustainability of a given fiscal rule from small-sample analysis of debt ratio trends.

A measure of the speed of convergence of the debt ratio to its long run value is given by the number of periods T* required before the probability of observing 'many' realizations which are 'far away' from the long-run trend becomes zero. The argument can be made precise by considering the probability of observing an infinite sequence of debt ratios which diverge from the deterministic trajectory by more than a given distance. For illustration, we shall focus on the case in which only output is stochastic, while the interest rate is fixed. Generalizations of this basic case can be obtained readily.

Let σ^2 be the variance of $\ln v_t$, 1/ the logarithm of the output shocks. By the law of iterated logarithm 2/ we have

(23) $\Pr(|\Sigma_t \ln v_t|/T>(1+\epsilon)b_T$, infinitely often) = 0

for any $\epsilon>0$, where $b_T=(2\sigma^2\ln(\ln T)/T)^{1/2}$, and for T>2. This means that the sample average of the (logarithm of the) technology shocks cannot differ from zero by more than $(1+\epsilon)b_T$, more than a finite number of times. Thus, b_T is the rate of convergence of the sample average to the distribution mean. Since the number of future realizations of the system is infinite, the number of deviations from the deterministic trend of the debt ratio is small, relative to all possible events over the time horizon. By applying the law of the iterated logarithm, one can appraise the speed at which the asymptotic tendency of d_T is likely to emerge as time elapses.

1/ The expected value of $\ln v_t$ is normalized to zero. See footnote 1, page 6. We are implicitly assuming stationarity of the increments of the output process. An equivalent argument would hold if the output process were stationary in levels, obviously a much more unrealistic assumption.

2/ See, for instance, Woodrofe (1975), Chapter 11.

Under Ponzi financing,

$$(24) \quad d_T = d_0 \exp[T\ln(\frac{1+r}{1+g}) - \sum_{t=1}^{T}\ln v_t]$$

Since the sequence $\Sigma_t \ln v_t/T$ remains inside the band $\{\pm b_T\}$ infinitely often with probability one, the sequence d_T will also remain inside the band $\{\bar{d}_T, \underline{d}_T\} = d_0\exp\{T[\ln((1+r)/(1+g)) \pm b_T]\}$ infinitely often with probability one. The behavior of a typical band $\{\bar{d}_T, \underline{d}_T\}$, corresponding to an asymptotically stable debt ratio, is illustrated in Chart 1 for a value σ^2=0.0045 and a two percent growth rate-interest rate differential. Notice the larger bandwidth for low values of T, corresponding to the higher degree of uncertainty surrounding the behavior of the debt ratio over short horizons. Eventually, b_T vanishes as $T \to \infty$, so that d_T converges with probability one to its long-run value. Before the long-run tendency prevails, however, output variability may be strong enough to make long streaks of increasing debt ratios quite likely. This can be illustrated by considering the behavior of the band $\{\bar{d}_T, \underline{d}_T\}$ for g>r. We see, in particular, that $\bar{d}_T < d_0$ for

$$(25) \quad \ln(\frac{1+r}{1+g}) < \sum_{t=1}^{T}\ln v_t/T$$

Therefore, the band which plays the role of "attractor" for the debt ratio falls below the initial value of debt, even for long-run stable fiscal plans, only for $T>T^*$, where T^* solves

$$(26) \quad (2\sigma^2\ln(\log T^*)/T^*)^{1/2} = \ln(\frac{1+r}{1+g})$$

One can verify that T^* is negatively related to (1+r)/(1+g) and positively to σ^2, so that the speed of convergence is lower for processes characterized by high variance and for low output growth-interest rate differentials. The relation between T^* and the differential (g-r) is illustrated in Chart 2 for different values of the output growth variability (σ^2=0.0030, 0.0045 and 0.0060), and for different levels of the average output growth-interest rate differential. As the chart shows, unless the differential is fairly large, the speed of convergence is slow, while T^* increases rapidly with the variance of the process. For example, for a two percent (g-r) differential, the band falls below the initial value of the debt ratio only after 47 periods for a variance of 0.0045. The simulations considered in the next section suggest that the relevant "period" length is unlikely to be less than one year, and is probably close to two years, while the values considered in Chart 2 for σ^2 are a reasonable approximation of actual output variability in industrialized countries.

CHART 1

Convergence band of the debt ratio

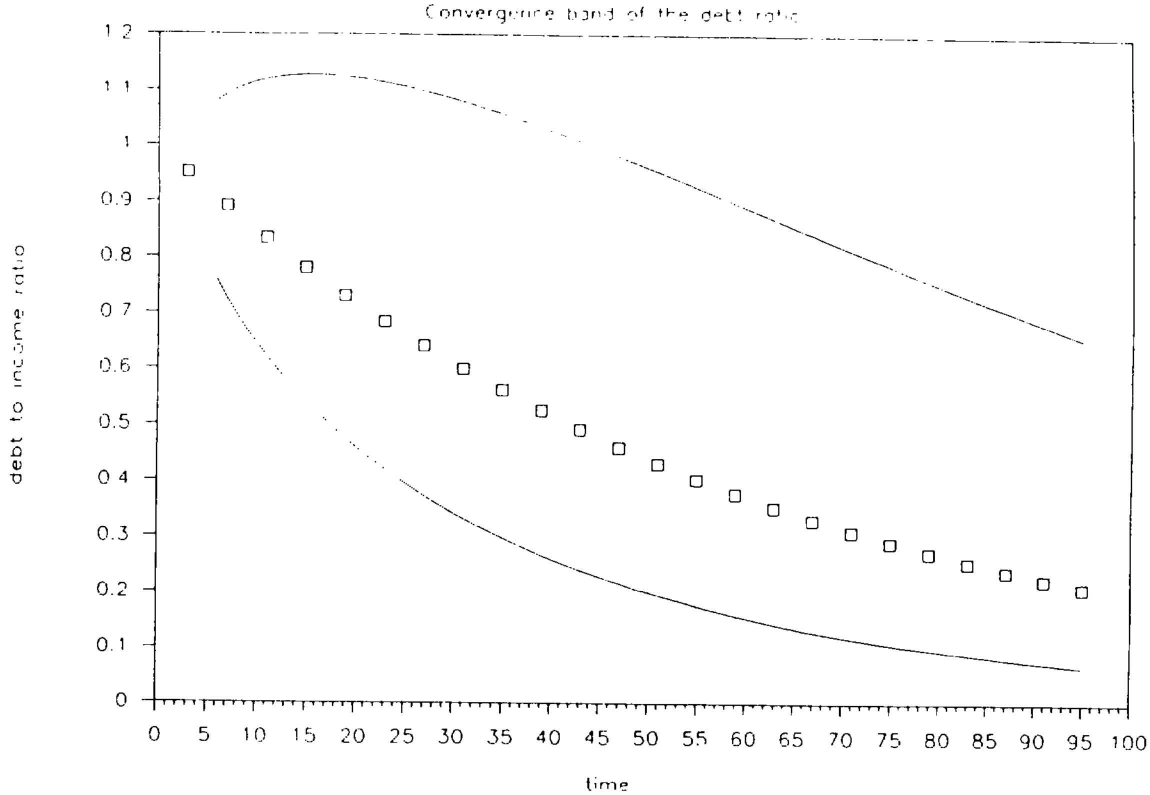
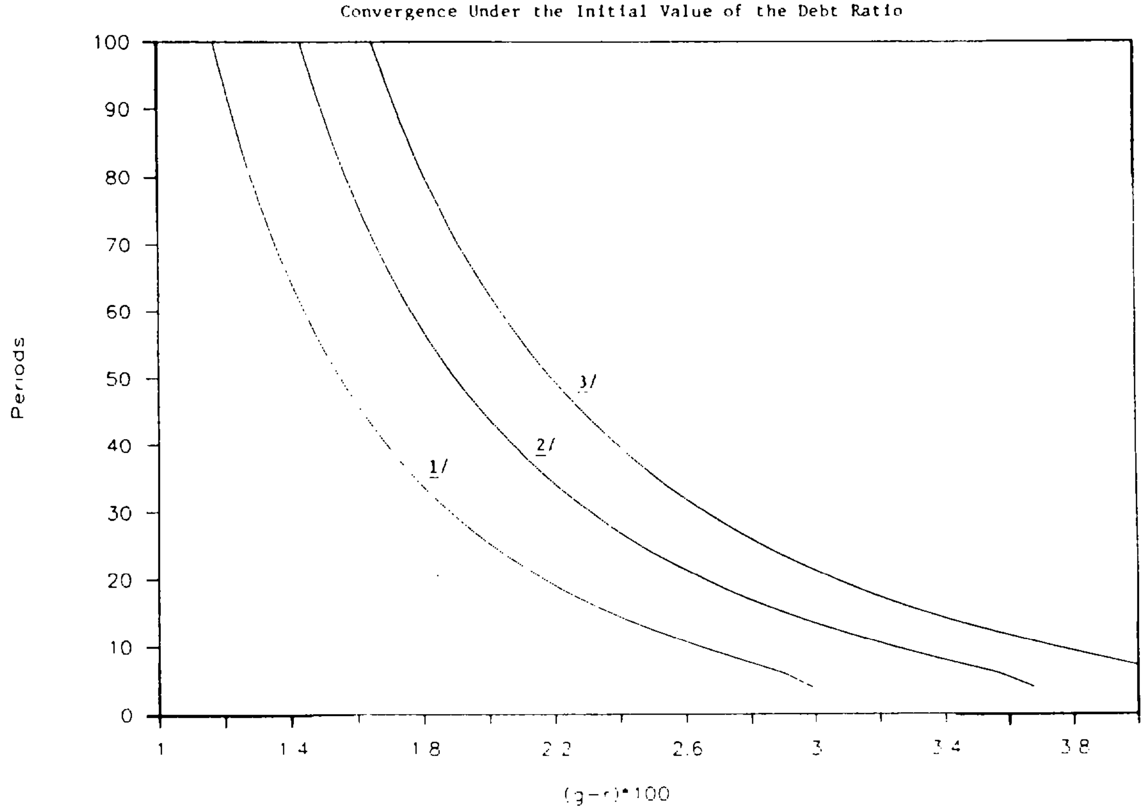


CHART 2

Convergence Under the Initial Value of the Debt Ratio



1/ Variance = .0030

2/ Variance = .0045

3/ Variance = .0060.

V. Numerical Simulations

In this section we take a closer look at the evidence on output and interest rate trends in the largest industrialized countries. While a thorough statistical analysis is beyond the scope of this paper, and the model that we use for our simulations is a rudimentary Lucas-type exchange economy model, our simple numerical simulations should help convey the practical relevance of the previous results, which nevertheless remain valid under quite general conditions.

The first issue that we consider is that of providing a reasonable lower bound to the interval used as a time unit in the previous analysis. In Table 1 we report the t-statistics computed for the null hypothesis that the autocorrelation coefficients of nominal GDP growth rates for the G-7 countries are zero at lags from 1 to 4; columns 1-4 refer to annual data, while columns 6-9 contain the same computation for two-year intervals. Starred values indicate correlation coefficients significantly different from zero at the 5 percent level.

For all countries considered, with the exception of the United States, the first order correlation is significantly positive, so are some of the second- and third-order correlations, while most of the fourth-order correlations are insignificantly different from zero. Our crude test thus suggests that the simple growth model with independent and identically distributed shocks, used in Sections III and IV, can be applied on intervals certainly no shorter than one year, and more likely close to two years. This conclusion is confirmed by examination of the t-statistics computed on two-year intervals (columns 6-9): in some cases the first-order correlation still appears significantly different from zero; in none of the cases considered can the hypothesis of zero second-order autocorrelation be rejected at the 5 percent significance level. Notice, furthermore, that the average value of σ^2 computed on two-year periods is about 0.45 percent, which corresponds to the central value of the range considered in Chart 2. In this case, the long-run stability implied by a two point (g-r) differential (or a one point differential on an annual basis) assures that the band which plays the role of "attractor" for the debt ratio falls under the initial level of debt only after about 47 periods (or 94 years).

To obtain more precise indications on the probability of observing debt ratios greater than a certain threshold at a given date, specific assumptions on the distribution of the productivity shocks are necessary. Again, to keep the analysis simple, we consider the simple case where output follows a lognormal process and the interest rate is non-stochastic. This case is consistent with a utility function with constant relative risk aversion and a Lucas-type exchange economy. It allows us to perform direct sensitivity tests on the level of the interest rate necessary to assure

Table 1. Correlation of GDP Growth Rates 1/

	One-Year Interval					Two-Year Interval				
	t-1	t-2	t-3	t-4	σ^2 2/	t-1	t-2	t-3	t-4	σ^2 2/
United States	1.38	0.87	1.55	0.73	0.05	1.31	0.58	0.15	-0.85	0.12
Japan	3.91*	2.87*	2.66*	2.49*	0.25	2.27*	1.68	0.99	-0.08	0.95
Germany	2.89*	0.93	0.72	0.72	0.07	1.99	0.35	0.59	0.39	0.19
France	3.92*	2.95*	2.23*	1.39	0.07	2.35*	0.76	-0.36	-0.95	0.23
United Kingdom	3.75*	2.95*	2.20*	1.79	0.19	2.62*	1.26	-0.30	-1.40	0.64
Italy	2.59*	1.74	1.79	0.46	0.28	1.65	0.05	-0.64	-1.17	0.83
Canada	2.47*	1.72	0.90	0.58	0.11	1.75	0.39	-0.10	-0.76	0.28

Source: IMF, International Financial Statistics.

1/ The table reports the t statistics computed for the autocorrelation coefficients computed at different lags. The sample period is 1961-90 except for Italy (1969-90) and Japan (1966-90). Starred t statistics indicate a correlation coefficient significantly different from zero at the 5 percent level.

2/ Estimated variance of the deviation of growth rates from average growth (variance of v_t in equation (11); in percent).

Ponzi-solvency, given the average rate of growth and variability of output. 1/ In this case, output follows equation (11), here reproduced for convenience:

$$(11) \quad Y_t/Y_{t-1} = (1+g)v_t$$

where $\ln v_t$ is a standard normal variable. The probability that after T periods of Ponzi financing the debt to output ratio is above a threshold of λ times the initial value d_0, is given by

$$(27) \quad P\{d_T>\lambda d_0\} = P\{ \sum_{t=1}^{T} \ln v_t/T < \ln[(1+r)\lambda^{-1/T}/(1+g)] \}$$

Since, $\Sigma_t \ln v_t/T \sim N(0,\sigma^2/T)$, the probability in (27) can be computed for each r, g, λ and T. For illustration, we report in Chart 3 the probability that the debt ratio be greater than 100 percent (upper panel) or greater than 150 percent (lower panel) of its initial value after T periods. Each period has a conventional length of two years (see the above discussion of Table 1); the output growth rate and the variance of the shocks are set at the average level for G-7 countries computed for two-year periods between 1961 and 1990, 2/ 22.5 percent and 0.45 percent, respectively. The four curves depicted in each panel correspond to growth rate-interest rate differentials of 0.1, 0.5, 1 and 2 points, on an annual basis.

Since all curves correspond to positive growth rate-interest rate differentials, we know that in the long run the debt to income ratio will converge to zero. Convergence is however a rather slow process and, as the top panel of the chart shows, even for the largest differential (2 points), the probability that the debt ratio will be greater than its initial value after 20 years is rather high (6 percent). After 50 years the probability is about 1, 11, 27, and 45 percent for differentials of 2, 1, 0.5, and 0.1 points, respectively. After 100 years it is still above 4 percent for a differential of 1 point, above 19 percent for a differential of 0.5 points, and above 43 percent for a differential of 0.1 points.

The probability of observing a ratio larger than 1.5 times its initial value at time T (lower panel) is of course smaller, but not insignificant. Indeed, the probability initially increases with time: this is because the effect of allowing more time for the debt ratio to reach values far away from its initial level initially dominates the effect of its stable deterministic component. For the lowest interest rate-growth rate differential (0.1 points) the probability of being 1.5 times above the initial value still increases after 200 years (and is around 20 percent). Even for a relatively high differential (1 point) the probability of

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1/ In a similar spirit, Bohn (1990) considers the probability of realizing long (possibly infinite) streaks of primary deficits while maintaining solvency in a dynamically efficient economy.

2/ 1969-90 and 1966-90, respectively, for Italy and Japan.

reaching a value of 1.5 times the initial level is about 1 percent after 50 years. 1/

The low speed of convergence of the debt to output ratio is important for two reasons. First, it may explain why--even when the average growth rate of the economy exceeds the average interest rate on debt--permanent deficits can be unsustainable. Even if the debt ratio converges in the long run, there is a positive (and indeed a relatively high probability) that in a finite span of time the debt ratio hits a threshold which may trigger a financial crisis. Recent models of fiscal sustainability (see for instance Giavazzi and Pagano (1989) and Alesina, Prati and Tabellini (1989)) do indeed link the probability of a financial crisis to the level of debt. Thus, maintaining primary surpluses as an insurance against unlucky income streaks may be necessary even if the debt ratio is asymptotically stable. In this sense, *almost sure* (asymptotic) convergence may be not sure enough.

Secondarily, the possibility of observing long sequences of increasing debt ratios, even for asymptotically stable paths, undermines the relevance of observed trends of the debt ratio for assessment of fiscal sustainability. Recent studies have observed the lack of empirical correlation between trends of the debt ratio and other indicators of fiscal sustainability (see Horne (1991) and Corsetti and Roubini (1991)). Our study provides a justification for the lack of correlation in a stochastic framework based on a simple criterion of solvency.

VI. Concluding Remarks

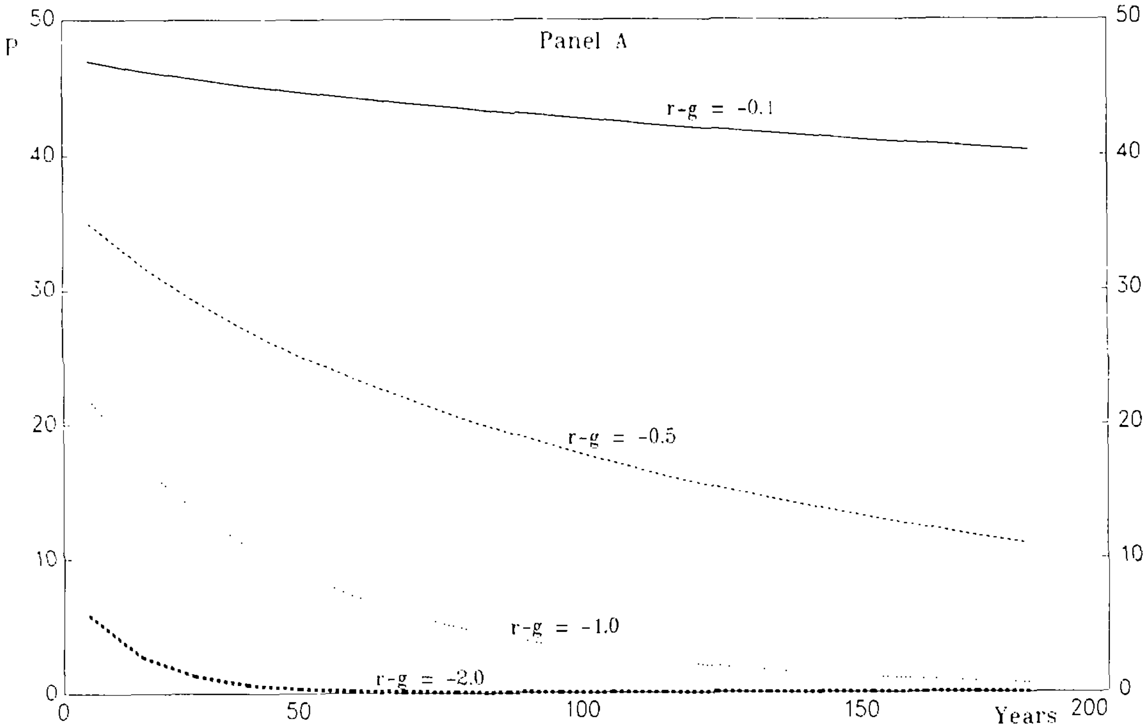
In this paper we have re-examined the implications for public debt dynamics of fiscal rules in which the government plays a Ponzi game in a stochastic environment. Our goal was to assess the extent to which the existence of aggregate uncertainty may reduce the scope for government Ponzi games, when the criterion for government solvency is that the debt to income ratio remains bounded in the long run. To this extent, the conclusions of our study can be easily summarized: stochastic economies do not behave in a substantially different way from deterministic economies in the long-run, and explosiveness of the expected debt ratio is no ground for rejection of a fiscal plan; however, for realistic parameter values, stochastic economies may frequently exhibit short-run trends and fluctuations which badly misrepresent their long-run tendencies. These deviations may hinder the feasibility of debt Ponzi games if the possibility of a financial crisis or some other hard bound on the debt ratio exists. They also undermine the usefulness of debt ratios as indicators of fiscal sustainability, for these ratios may likely exhibit long (in our examples: decade-long) streaks of increasing values and still be consistent with debt stability in the long run.

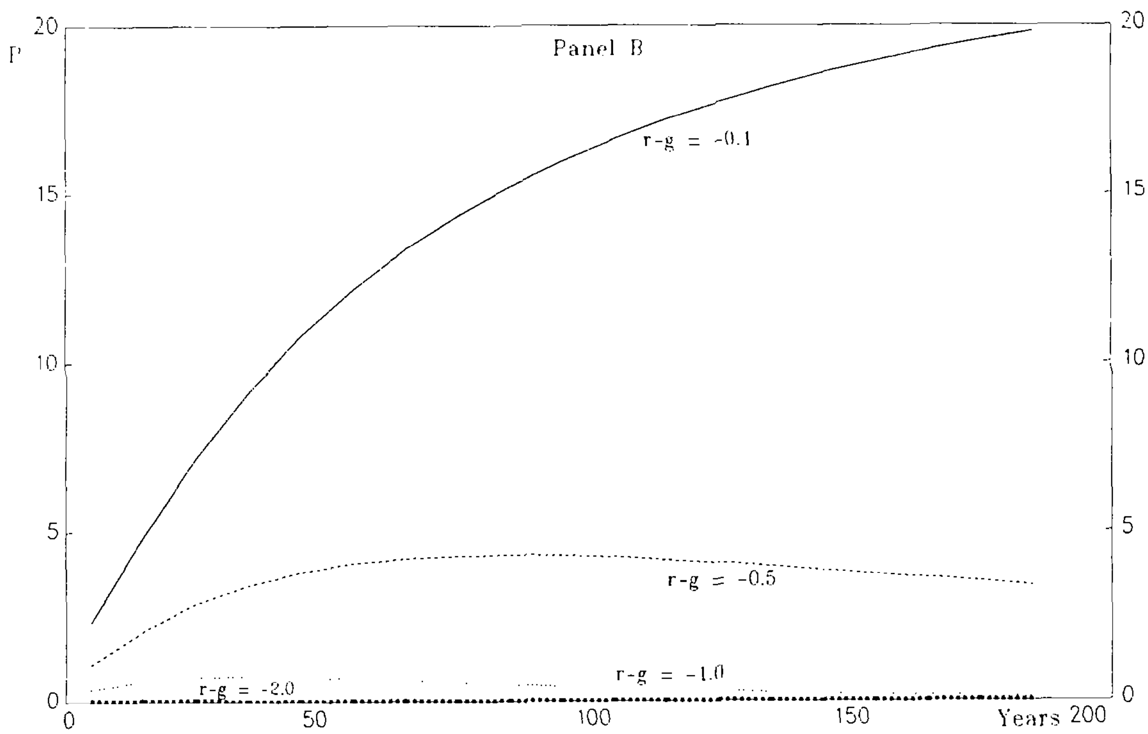
1/ Of course these probabilities would increase in the presence of a permanent primary deficit. Recall that we are considering the case of a 'pure' Ponzi game, with a once-for-all initial deficit followed by a subsequently balanced primary balance.

CHART 3

PROBABILITY OF EXCEEDING A GIVEN CEILING 1/

(In percent)





1/ Probability that after T years the debt to output ratio is above the initial value (Panel A) and 1.5 times the initial value (Panel B).

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