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Estimation of the Near Unit Root Model of Real Exchange Rates

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Abstract

The time-series properties of real exchange rates, on a number of definitions, for 22 industrial countries during 1979-95 were used to re-examine whether PPP holds. It is shown that if real exchange rates reverted to a constant mean slowly, say by five percent a month, then at standard levels of significance we should expect 11 of the 22 series examined to yield evidence of mean reversion and to reject that hypothesis of a unit root. Using models that imply a constant unconditional mean or trend-stationary productivity changes, we find that only one of the 22 real exchange rates shows evidence against unit roots. This low rate of rejection of unit roots in real exchange rates can be construed as evidence against PPP.

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C22, F31

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Table of Contents

Summary

The theory of purchasing power parity (PPP) states that real exchange rates should converge to an equilibrium level at which a unit of one currency should buy the same basket of goods in any country. If exchange rates do not converge to PPP levels, large--and in some cases persistent--swings can occur in a country's international competitiveness, and international comparisons of real income based on market exchange rates will be misleading.

There is a very large empirical literature on tests of the PPP hypothesis, which on the whole finds little evidence to support it. But recent developments in time-series econometrics--in particular, advances in the methods of testing for the stationarity of variables (unit root tests)--have led to a revival of interest in new empirical tests of PPP theory. This revival has focused largely on whether real exchange rates have unit roots--that is, whether variations in real exchange rates are bounded. If real exchange rates can be characterized by a unit root, PPP does not hold because there is no propensity to revert back to any equilibrium level.

A problem that arises with unit root tests is that they have low power in many cases of empirical interest, especially when shocks dissipate slowly. In such cases, the failure to reject unit roots cannot be construed as evidence against PPP, since there are insufficient data to observe that PPP holds even if it in fact does. This problem can be overcome by using additional information. In this paper, the additional information required to draw conclusions about the relevance of PPP is obtained by using interval estimators--which, unlike hypothesis tests, do provide information when a hypothesis is not rejected--and by extending the analysis to cover many countries.

The time-series properties of real exchange rates, on a number of definitions, for 22 industrial countries during 1979-95 were used to re-examine whether PPP holds. If real exchange rates revert to a constant mean slowly--say, by 5 percent a month--then it is shown that at standard levels of significance one should expect 11 of the 22 series examined to yield evidence of mean reversion and reject unit root tests. Analysis using models that imply a constant unconditional mean or trend-stationary productivity changes finds that only 1 of the 22 real exchange rates shows evidence against unit roots. The lower-than-expected rate of rejection of unit roots in real exchange rates can be construed as evidence against PPP.

I. Introduction

Meese and Rogoff (1983) demonstrated that a variety of linear structural exchange rate models failed to forecast more accurately than a naive random walk model for both real and nominal exchange rates. If the real exchange rate follows a random walk then innovations to the real exchange rate persist and the time series can fluctuate without bound. This result is contrary to the theory of Purchasing Power Parity (PPP) which at its most basic level states that there is an equilibrium level to which real exchange rates converge such that a unit of one currency should buy the same basket of goods in any country.

The conclusion of Meese and Rogoff (1983) has been reached by numerous authors since the publication of their paper. Typically, this conclusion was derived by using formal statistical tests that failed to reject the null hypothesis of a unit root against the alternative of a stationary autoregressive (AR) model. An explanation of this result often given is that unit root tests have low power because of the relatively short sample periods under study. The recent empirical literature on testing the existence of PPP has developed in tandem with recent developments in the unit root econometrics literature.[1] The contribution of this paper will continue this trend by showing that the latest developments of unit root econometrics can be used to address the low power problem. Perhaps more importantly, the paper will also address the problem of near unit root bias that is typically ignored in the empirical literature and that biases empirical results in favor of finding PPP.

In an attempt to overcome the low power problems inherent in classical unit root testing procedures Schotman and van Dijk (1991) use a Bayesian analysis to examine whether real exchange rates contain a unit root. The results of the their Bayesian analysis is not without controversy. There is the question of choosing the most appropriate prior to use. Schotman and van Dijk use a flat prior in an attempt to obtain a degree of impartiality but there is still a question of whether a prior is really noninformative (Phillips, 1991). The introduction of impartiality properties is desirable, especially in the real exchange rate case where the magnitude of the autoregressive parameter is a contentious issue. However, classical hypothesis tests and proper Bayes estimators do not do this.

Another problem in using the autoregressive or unit root model to analyze the persistence of shocks in the real exchange rates is that standard estimators, such as least squares, are significantly downwardly biased in models that contain constants or time trends, especially when the autoregressive parameter is close to but less than one. This bias offsets the low power properties of unit root tests. For example, in an AR(1) model with a coefficient equal to $\cdot 95$, a constant, a sample size equal to 200, and normal errors,

[1] For surveys on PPP and exhange rate economics see Dornbusch (1991), Frankel (1993), Isard (1995), and Taylor (1995). For surveys of recent evidence on PPP see Boucher Breuer (1994), Froot and Rogoff (1995), and MacDonald (1995).

the unit root model can be rejected if the estimated coefficient is less than ·93.[2] The median of the least squares distribution of the autoregressive coefficient is ·93. Thus, in roughly fifty percent of cases the unit root model will be correctly rejected.[3]

In the case of the real exchange rate, where the failure to reject the null hypothesis cannot be construed as providing evidence in favor of the null, additional information is required. Point and interval estimators are useful statistics for providing such information and will be used in this paper. In addition, we will generate a transformation from some initial estimator to an (asymptotically) median-unbiased estimator in order to correct the bias noted above. Median unbiased estimators for autoregressive models have been proposed by Andrews (1993), Andrews and Chen (1994), McDermott (1994), Rudebusch (1993), and Stock (1991). The type of initial estimators considered in this paper were first proposed by Phillips (1987*a*) and Perron (1988) in the context of unit root testing. These estimators allow for a wider class of error processes than previously considered in the "median-unbiased" literature. To achieve the objectives outlined above, existing results in the time series literature on local-to-unity asymptotics will be exploited whenever possible. For this purpose, the papers by Phillips (1987*b*) and Phillips and Perron (1988) are particularly pertinent. The median unbiased estimators employed in this paper are better suited than alternative median unbiased estimators for modelling real exchange rates. This is because the estimators used here are based on initial estimators that are robust to weakly dependant and heterogeneously distributed time series.

Median unbiased estimates and 90% asymptotic confidence intervals of the autoregressive coefficient of an AR(1) model were computed using real exchange rates from a group of 22 industrial countries. The analysis was first conducted using an AR(1) model with a constant since the basic theory of PPP imposes the restriction that real exchange rates have a constant unconditional mean. A stationary AR(1) model with a constant is consistent with this restriction whereas a unit root model is not. An AR(1) model with a time trend was also used since the real exchange rate may move if there are differing productivity trends across countries. This adjusted theory is consistent with a trend stationary model if productivity changes are trend stationary. Since productivity changes move slower than changes in exchange rates, this assumption is reasonable.

Three different measures of the real exchange rate were employed to see if the results were sensitive to the types of indices used. These measures where: (i) the Real Effective Exchange Rate (REER) based on consumer prices, (ii) the REER based on normalized unit labor costs, and (iii) real bilateral exchange rates based on stock market prices. The REERs were used because they provided a reasonably comprehensive and comparable series based on movements in costs and prices. The real rates based on stock market prices were used because such prices can move as fast as nominal exchange rates.

[2]The 5 percent critical value of the test statistic $T(\widehat{\alpha} - 1)$ in Fuller (1976) is $-13{\cdot}9$.

[3]A selective survey of some recent theoretical work and remaining problems of unit root econometrics covering the case when the autoregressive root is large but not exactly one, can be found in Stock (1995).

The stickiness of goods and labor market prices may generate long-term persistence in real exchange rates. Since equity prices are not sticky and stock market shares can and are traded virtually continuously, then such prices maybe more appropriate for measuring real exchange rates, especially when one is using high frequency data. One problem with using equity price data is if there are persistent divergent expectations of future returns across national equity markets then real exchange rates measured this way will contain a unit root automatically. This problem will be mitigated since financial markets have become more integrated over the last decade.

Finally, having obtained the estimates of the autoregressive coefficient for the countries individually, the information is pooled by estimating the density of the autoregressive coefficient, treating each country as a single observation. If the data generating process for real exchange rates is common across countries then the estimated density of the autoregressive coefficient will provide a useful summary measure of the results that is easy to interpret. In a related paper, Wu (1996) uses panel-data methods to pool cross-country information and then tests for unit roots. However, this method is still subject to the median bias that favors rejecting unit roots.

The remainder of the paper is as follows. The near unit root model and the initial estimators used to provide point estimates are presented in Section II, while the median unbiased estimators themselves are explained in Section III. The assumptions underlying the initial estimators are presented in an appendix. A Monte Carlo experiment is conducted in Section IV in order to evaluate the performance of the median unbiased estimator. For the reader only interested in the empirical results, these sections may be skipped. The empirical results are contained in Section V and the concluding remarks are contained in Section VI.

II. The Near Unit Root Model

Three regression models of the univariate time series $\{y_t : t = 0, ..., T\}$ are considered in this section and are defined as follows:

$$\begin{array}{llcll}
\text{Model 1:} & y_t & = & \alpha y_{t-1} + u_t, & t = 1, ..., T; \\[1em]
\text{Model 2:} & y_t^* & = & \alpha y_{t-1}^* + u_t, & \\
 & y_t & = & \mu + y_t^*, & t = 1, ..., T; \\[1em]
\text{Model 3:} & y_t^* & = & \alpha y_{t-1}^* + u_t, & \\
 & y_t & = & \mu + \beta \frac{t}{T} + y_t^*, & t = 1, ..., T.
\end{array}$$

The assumption on the autoregressive coefficient is given below.

Assumption 1

$$\alpha = \exp(c/T), \text{ } \textit{where c is a fixed constant.} \tag{1}$$

Formally, time series generated by Models 1 to 3 constitute a triangular array of the type $\{y_{Tt} : 1 \le t \le T;\ T \ge 1\}$ since the autoregressive coefficient depends on T. However, this formality is not essential and for simplicity we shall denote y_{Tt} by y_t.

We could re-write equation (1) in Assumption 1 as $\exp(c/T) = 1 + c/T + O(T^{-2})$. In this form (1) is analogous to the Pitman drift assumption of asymptotic power studies. Assumption 1 allows us to utilize the local-to-unity asymptotic theory described in Phillips (1987*b*). When $c < 0$ Models 1, 2, and 3 will be stationary. When $c > 0$, Models 1, 2, and 3 exhibit explosive behavior. The case that has attracted a great deal of attention recently is when $c = 0$, the unit root case.

We now define the estimators to be considered in this paper. Let $\hat{\alpha}_i^+$ be the estimator of α for regression Model i. A suitable form of such estimators is suggested by Phillips (1987*a*) and Phillips and Perron (1988) and is given by

$$\hat{\alpha}_i^+ = (Y_{-1}'(I - P_i)Y_{-1})^{-1}(Y_{-1}'(I - P_i)Y - T\hat{\lambda}_i), i = 1,2,3; \tag{2}$$

where $Y = (y_1, ..., y_T)'$, $Y_{-1} = (y_0, ..., y_{T-1})'$, $P_1 = 0 \in \mathbf{R}^{T\times T}$, $P_i = X_i(X_i'X_i)^{-1}X_i'$, i = 2,3; $X_2 = (1, ..., 1)' \in \mathbf{R}^T$, $X_3 = (X_2|\ x_3) \in \mathbf{R}^{T\times 2}$, $x_3 = (1, 2, ..., T)'$, $\hat{\lambda}_i = (1/2)(s_{Tb}^2 - s_u^2)$, i=1,2,3. To compute the estimators $\hat{\alpha}_i^+$ (i=1,2, and 3) we first need an estimator of the long-run variance of u_t, s_{Tb}^2 and an estimator of the variance of u_t, s_u^2. Such estimators are defined below

$$\begin{aligned} s_{Tb}^2 &= T^{-1}\textstyle\sum_{t=1}^T \hat{u}_t^2 + 2\,T^{-1}\sum_{\tau=1}^b w\left(\frac{\tau}{b+1}\right)\sum_{t=\tau+1}^T \hat{u}_t\hat{u}_{t-\tau}, \\ s_u^2 &= T^{-1}\textstyle\sum_{t=1}^T \hat{u}_t^2, \end{aligned} \tag{3}$$

where $\hat{u}_t$'s are the estimated residuals from the OLS regression of Y on Y_{-1}, or Y on (X_1, Y_{-1}), or Y on (X_2, Y_{-1}), $w(\cdot)$ is a real-valued kernel defined below, and b is a bandwidth parameter.

For the empirical application to follow, the quadratic spectral kernel will be employed. This kernel has the form

$$w(x) = \frac{25}{12\pi^2 x^2}\left(\frac{\sin(6\pi x/5)}{6\pi x/5} - \cos(6\pi x/5)\right). \tag{4}$$

The prewhitening technique of Lee and Phillips (1993) is also employed to estimate the long-run variance s_{Tb}^2. The idea of using prewhitening to improve estimation of covariances was introduced by Andrews and Monahan (1992), who used an AR(p) model with fixed p to develop a prewhitened kernel estimator. Lee and Phillips (1993) modified the Andrews and Monahan (1992) procedure by using a data-based ARMA model to develop a prewhitened kernel estimator.

For empirical work, it is useful to have a means of choosing the bandwidth in an automatic way so that comparisons between different studies can be made on the basis of a standardized method. The operational estimator of b for the automatic selection procedure is

$$\hat{b}+1 = \left[\frac{q\, w_q^2 \hat{\phi}(q) T}{\int w^2(x)dx}\right]^{1/(2q+1)}, \tag{5}$$

where $\hat{\phi}(q) = 4\hat{\rho}^2/(1-\hat{\rho})^4$, $\hat{\rho} = \hat{U}'\hat{U}_{-1}/\hat{U}_{-1}'\hat{U}_{-1}$, $\hat{U} = (\hat{u}_1, \ldots, \hat{u}_T)'$, $\hat{U}_{-1} = (\hat{u}_0, \ldots, \hat{u}_{T-1})'$, $w_q \equiv \lim_{x\to 0}(1-w(x))/|x|^q$. The type of kernel employed determines the value of q. For the quadratic spectral kernel, $q = 2$. $\hat{\phi}(q)$ may take several other forms depending on the model employed to describe the dependence in the $\hat{u}_t$'s.

III. Median-Unbiased Estimators

We now outline a procedure for computing an asymptotically median-unbiased estimator of α that is based on the serial correlation corrected estimator $\hat{\alpha}^+$, and for obtaining confidence intervals for such an estimator. Median-unbiased estimators for autoregressive models have been proposed by previous authors. Andrews (1993) proposed an exactly median-unbiased estimator with exact confidence intervals for AR(1) models. This method has been extended by Andrews and Chen (1994) for the sum of the coefficients in AR(p) models. However, in this case the estimator is only approximately median-unbiased. McDermott (1994) presents median unbiased estimators and confidence intervals for the case when there are structural breaks in the time series. Rudebusch (1993) also examines median-unbiased estimators for autoregressive models using simulation methods. Stock (1991) considered an asymptotic confidence interval of the largest autoregressive root in a time series.

The objective in this section is to produce a median-unbiased estimator (and confidence intervals) for the autoregressive coefficient in Models 1, 2, and 3 based on the local-to-unity asymptotic theory and allowing the assumptions on the error structure (described in the appendix) to be more general than those used in previous work. For example, Andrews (1993) assumes iid errors while Andrews and Chen (1994) assume AR(p) errors. Neither paper allows for the possibility of a MA error structure (unless $p \to \infty$ as $T \to \infty$).

Below we discuss the asymptotically median-unbiased estimator and the asymptotic confidence interval. Suppose $\hat{\theta}$ is an estimator of a scalar parameter whose median function $m(\theta)$ is uniquely defined and is strictly increasing on the parameter space Θ which is a finite interval. Then $\hat{\theta}_u$ is a *median-unbiased estimator* of θ, where $\hat{\theta}_u = m^{-1}(\hat{\theta})$ and $m^{-1}(\cdot)$

is the inverse function of $m(\cdot)$ that satisfies $m^{-1}(m(\theta)) = \theta$ for $\theta \in \Theta$. In addition, suppose $\hat{\theta}_i$ is an estimator whose p_1 and p_2 quantiles are uniquely defined, depend asymptotically only on θ, and are strictly increasing in θ on the parameter space Θ. Let $q_{p_1}(\theta)$ and $q_{p_2}(\theta)$ denote these quantile functions. Then, an asymptotic $100(1-p_1-p_2)\%$ confidence interval for θ is given by $[\hat{L}, \hat{U}]$, where $\hat{L} = q_{p_2}^{-1}(\hat{\theta})$, $\hat{U} = q_{p_1}^{-1}(\hat{\theta})$ and for $j = 1, 2$, $q_{p_j}^{-1}(.)$ is the inverse function of $q_{p_j}(.)$ that satisfies $q_{p_j}^{-1}(q_{p_j}(\theta)) = \theta$ for $\theta \in \Theta$.

Below we describe how to construct asymptotic local-to-unity central confidence intervals by Monte Carlo simulation. The asymptotically median-unbiased estimator $\hat{\alpha}_{AMU}$ of α and the central confidence interval $[\hat{L}, \hat{U}]$ for α of asymptotic confidence level $100(1-p_1-p_2)\%$ are defined by

$$\hat{\alpha}_{AMU} = 1 + c_{MED}/T, \tag{6}$$

and

$$[\hat{L}, \hat{U}] = [1 + c_L/T, 1 + c_U/T]. \tag{7}$$

To apply this method we first compute α_i^+ and then find c_{MED}, c_L and c_U via the functions $m_\alpha^{-1}(.)$ and $q_\alpha^{-1}(.)$. These functions have been tabulated by simulation and are reported in Table 1 for c ranging from -200 to 10 and displayed in Charts 1, 2, and 3 for c ranging from -40 to 10.[4] The method works because the limit distributions of the estimators $\hat{\alpha}_i^+$ $(i = 1, 2, 3)$ (given in Result 1 in the appendix) depend only on c, are increasing in c, and are continuous in c. This is in contrast to the limit distribution of the least squares estimator of α from the Dickey-Fuller regression which also depend on all the stable roots in the lag polynomial that describe the short-run dynamics. Using the estimators $\hat{\alpha}_i^+$ $(i = 1, 2, 3)$ avoids this problem by eliminating the nuisance parameter dependence via the nonparametric adjustment term $\hat{\lambda}$.

The median and quantile functions depicted in Charts 1, 2, and 3 are close to being strictly increasing in c although a small region where these function are not increasing can be detected around $c = -2$. The functions can be smoothed in this region such that the resulting functions are strictly increasing. The kink at $c = 0$ in Charts 2 and 3 shows that the quantiles of the estimators $\hat{\alpha}_i^+$ (i=2,3) of α from the autoregressive models have a discontinuous behavior between the stationary and nonstationary regions of the model. This kink is most pronounced in the model with a time trend. The extent of the bias in the estimator can also be observed. Consider model 3 in the unit root case ($c = 0$). The median of the estimator is ·9 and the ·95 quantile is less than unity. Thus the 90% central

[4]The limit distribution used to construct the median and quantile functions was evaluated by Monte Carlo simulation for $T = 100$ with 1,000 replications. The simulated data was generated from the autoregressive model $y_t = \alpha y_{t-1} + \epsilon_t$, where ϵ_t is iid $N(0, 1)$, $\alpha = 1 + c/T$ and $y_0 = \epsilon_0/(1-\alpha^2)^{0.5} 1(c < 0)$. μ and β have been set to zero and σ^2 has been set to one since the distribution of $\hat{\alpha}_i^+ (i = 1, 2, 3)$ is invariant with respect to (σ^2, μ, β) (see Result 2 in the appendix).

Chart 1:
Median & Quantile Functions of Model 1

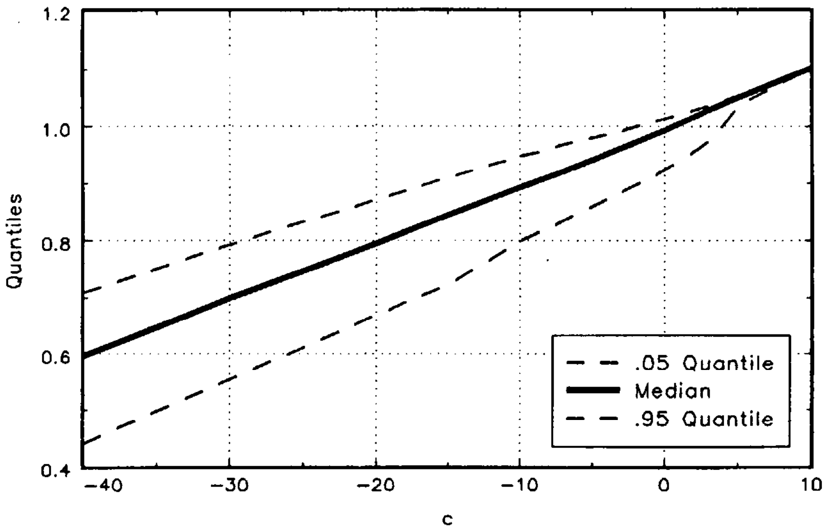

Chart 2:
Median & Quantile Functions of Model 2

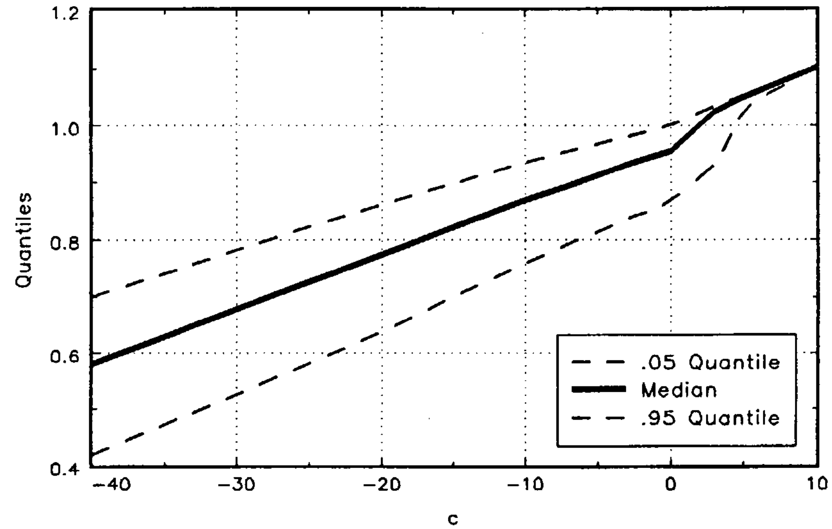

Chart 3:
Median & Quantile Functions of Model 3

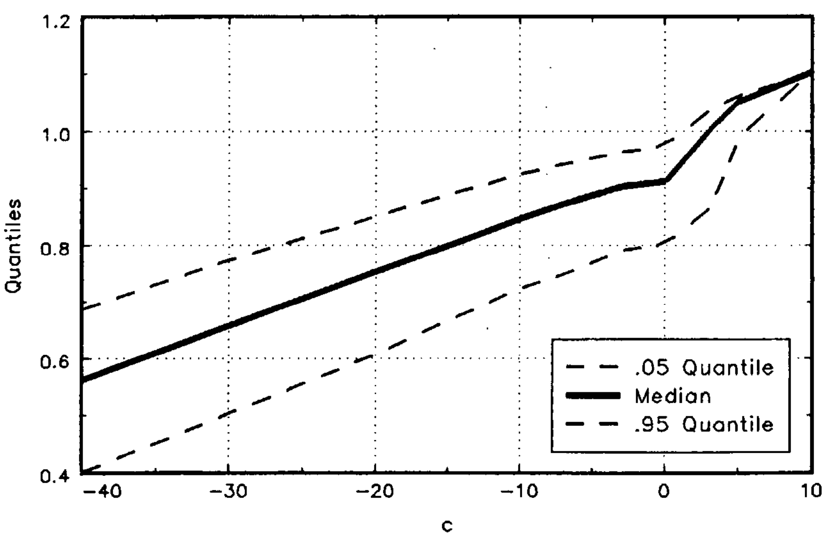

Chart 4:
Japan--Real exchange rates

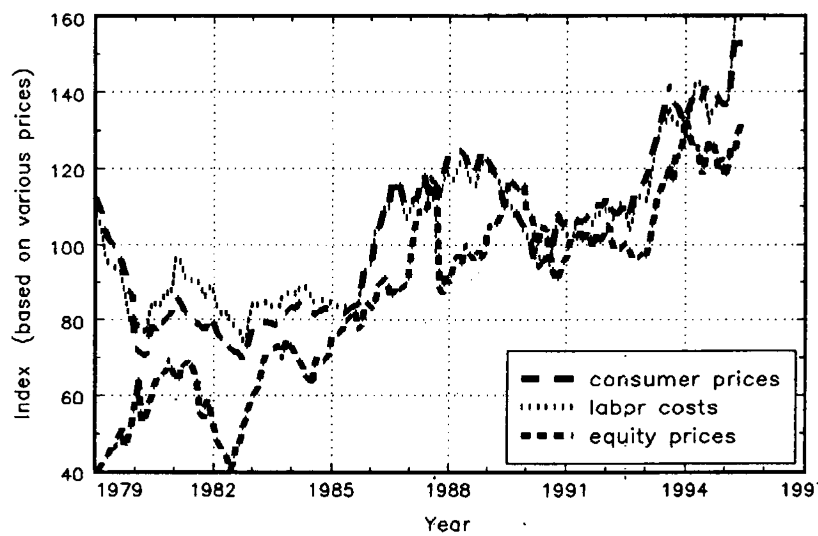

Table 1. Quantiles of the Estimators $\hat{\alpha}^{+}$

	$\hat{\alpha}_1^{+}$			$\hat{\alpha}_2^{+}$			$\hat{\alpha}_3^{+}$		
	.05	.5	.95	.05	.5	.95	.05	.5	.95
-200	-1.006	-.997	-.958	-1.006	-.997	-.956	-1.006	-.997	-.958
-180	-.875	-.792	-.669	-.878	-.794	-.675	-.878	-.797	-.680
-160	-.709	-.596	-.451	-.714	-.602	-.457	-.715	-.605	-.459
-140	-.532	-.395	-.248	-.539	-.404	-.253	-.541	-.409	-.262
-120	-.357	-.198	-.044	-.365	-.206	-.051	-.370	-.214	-.060
-100	-.167	.002	.154	-.180	-.009	.146	-.186	-.020	.137
-90	-.072	.101	.250	-.080	.087	.242	-.097	.076	.228
-80	.025	.201	.344	.015	.186	.336	-.002	.174	.323
-60	.232	.397	.528	.214	.383	.518	.197	.369	.506
-50	.334	.497	.618	.316	.481	.610	.298	.466	.596
-40	.441	.596	.708	.419	.580	.700	.399	.561	.686
-30	.553	.698	.792	.525	.678	.782	.501	.657	.773
-20	.669	.794	.870	.638	.774	.861	.606	.753	.849
-15	.723	.844	.910	.701	.821	.898	.666	.799	.888
-10	.797	.892	.947	.758	.868	.935	.723	.845	.923
-7	.834	.921	.968	.792	.894	.955	.748	.871	.941
-3	.884	.962	.992	.835	.931	.980	.790	.901	.963
0	.922	.992	1.013	.865	.956	1.000	.801	.911	.973
3	.974	1.029	1.036	.931	1.024	1.033	.863	1.002	1.036
5	1.036	1.050	1.053	1.030	1.049	1.052	.988	1.050	1.060
10	1.100	1.100	1.100	1.100	1.100	1.100	1.100	1.100	1.100

Notes: The limited distribution used to construct these quantiles were evaluated by simulation with 1,000 replications for the estimators $\hat{\alpha}_i^{+}$ (i=1,2,3). The simulated data were generated from the model $(1-\alpha B)y_t=\epsilon_t$, where ϵ_t is iid and N(0,1), $\alpha=1+c/T$, $Y_0=\epsilon_0/1-\alpha^2)^{0.5}1(c<0)$. The coefficient on the deterministic regressors have been set to zero and the variance of the innovations has been set to 1 since the distribution of the estimators $\hat{\alpha}_i^{+}$ (i=1,2,3) is invariant with respect to these parameters (see appendix for details).

confidence interval will cover the true value of unity less then 5 percent of the time suggest that one can make significant improvements to standard estimators of AR(1) models.

To illustrate the use of the tables, consider the estimator $\hat{\alpha}^+$of α for Model 1. If $T = 100$ and $\hat{\alpha}^+ = \cdot 962$ then Table 1 yields $c_{MED} = -3$, $c_L = -7$ and $c_U = 2\cdot 3$. Hence $\hat{\alpha}_{AMU} = \cdot 97$ and a 90% confidence interval $[\widehat{L}, \widehat{U}] = [\cdot 93, 1\cdot 023]$. Since the grid of c values given in Table 1 is finite, interpolation between values will often be required or one could compute by simulation the functions $m_{\alpha}^{-1}(.)$ and $q_{\alpha}^{-1}(.)$ for as fine a grid as required.

IV. Monte Carlo Analysis

In this section the finite sample properties of the estimators defined in (2) are examined. The generating process employed in this analysis is the nearly-integrated moving average model,

$$(1 - \alpha B)y_t = (1 - \theta B)\varepsilon_t, \tag{8}$$

where $\alpha = 1 + c/T$ and B is the backshift operator. We will examine (8) using the following parameter values: $c = (2, 0, -2, -5, -10)$ and $\theta = (\cdot 5, 0, -\cdot 5)$. When conducting the simulation the first distribution for ε_t considered was the standard normal N(0,1). Given that the quantile functions where derived using the standard normal, this distribution should give the best results. To analyze the robustness of the procedure to non-normality of the innovations four other distributions where considered: the Student's t with 3 degrees of freedom, Chi-squared with 4 degrees of freedom (shifted to have zero mean), Rademacher (± 1 with probability $\cdot 5$ each), and Cauchy. These distributions exhibit thick tails, skewness, discreteness, and extremely thick tails, respectively.

The results of the Monte Carlo analysis are reported in Table 2. The entries in this table are the fraction of times that the computed 90% confidence intervals contain the true value of α for the different experiments and thus the expected value of each entry is $\cdot 9$.

The sample coverage probabilities are close to their theoretical value of $\cdot 9$ except when $\theta = \cdot 5$. Thus we conclude that the asymptotic approximations perform well, except in the case when the moving average polynomial $\theta(B) = (1 - \theta B)$ nearly cancels the autoregressive polynomial $\alpha(B) = (1 - \alpha B)$, and even then, the higher the value of α the closer the sample coverage probabilities are to $\cdot 9$.

In this problematic case, the estimated confidence intervals tend to lie to the left of the true α. Producing confidence intervals to the left of the true value of α when θ is near α is not a serious problem for most purposes. For the AR(∞) representation of (8), which is $(1 - \sum_{i=1}^{\infty} \theta^{i-1}(\alpha - \theta)^i B^i)y_t = \varepsilon_t$, coefficients for lags greater than one will be close to zero (for example, when $\alpha = \cdot 9$ and $\theta = \cdot 5$ the coefficient on the second lag of the AR(∞)

Table 2. Monte Carlo Results for Finite Sample Coverage Probabilities for Local-to-Unity 90 Percent Asymptotic Confidence Intervals

Distribution of Errors	α	no constant θ 0.0	no constant θ 0.5	no constant θ -0.5	constant θ 0.0	constant θ 0.5	constant θ -0.5	time trend θ 0.0	time trend θ 0.5	time trend θ -0.5
Standard normal	.90	.877	.205	.852	.878	.171	.880	.882	.128	.900
	.95	.883	.394	.921	.881	.322	.926	.869	.206	.932
	.98	.899	.678	.923	.891	.505	.925	.880	.296	.926
	1.00	.892	.735	.894	.896	.559	.812	.877	.324	.906
	1.02	.892	.816	.895	.899	.755	.908	.813	.331	.864
Student's t with 3 df	.90	.883	.185	.863	.892	.159	.882	.901	.118	.910
	.95	.934	.484	.955	.919	.378	.882	.916	.247	.949
	.98	.941	.756	.946	.902	.544	.940	.883	.304	.911
	1.00	.897	.735	.895	.893	.572	.918	.885	.318	.910
	1.02	.940	.853	.950	.929	.782	.943	.882	.385	.919
Chi-squared with 4 df	.90	.881	.198	.852	.898	.157	.888	.896	.119	.910
	.95	.930	.498	.953	.925	.385	.943	.915	.246	.950
	.98	.935	.765	.946	.908	.564	.924	.878	.304	.910
	1.00	.882	.724	.889	.890	.544	.903	.891	.310	.914
	1.02	.934	.852	.943	.939	.785	.964	.871	.385	.907
Rademacher	.90	.873	.213	.850	.882	.161	.882	.889	.122	.909
	.95	.935	.533	.956	.915	.414	.936	.912	.260	.948
	.98	.955	.809	.952	.902	.597	.913	.879	.310	.906
	1.00	.892	.735	.895	.898	.554	.907	.881	.322	.912
	1.02	.939	.853	.942	.939	.789	.952	.873	.382	.914
Cauchy	.90	.936	.126	.910	.946	.089	.925	.938	.073	.932
	.95	.955	.473	.962	.951	.277	.956	.934	.193	.943
	.98	.948	.730	.948	.935	.504	.929	.898	.311	.903
	1.00	.914	.788	.902	.900	.643	.891	.905	.346	.909
	1.02	.933	.854	.933	.955	.835	.951	.915	.448	.923

Simulations for model with: no constant, constant, time trend.

Note: The Monte Carlo simulation was conducted for T=100 with 5,000 replications for the estimators $\hat{\alpha}_i^{+}$ (i=1,2,3), using the quantile functions $q_{.05}(\hat{\alpha}_i^{+})$ and $q_{.95}(\hat{\alpha}_i^{+})$ reported in Table 1. The entries are the fraction of times that the computed 90% confidence interval contains the true value of α. The simulated data were generated from the model $(1-\alpha B)y_t=(1-\theta B)\epsilon_t$, where $\alpha = 1 + c/T$ and $y_0 = \epsilon_0/(1-\alpha^2)^{0.5}1(c<0)$.

model is $\cdot 08$). Thus, (8) is equally well represented as an AR(1) model with a lower autoregressive coefficient than in the ARMA(1, 1) model. For example, when $\alpha = \cdot 9$ and $\theta = \cdot 5$ the model $y_t = \cdot 4 y_{t-1} + \varepsilon_t$ will generate similar forecasts. While we will underestimate the persistence of a particular shock when using the AR(1) model, the average contribution of each shock to the long-term path of y_t is small.

Qualitatively similar results were obtained using the four distributions, Student's t with 3 degrees of freedom, Chi-squared with 4 degrees of freedom (shifted to have zero mean), Rademacher (± 1 with probability $\cdot 5$ each), and Cauchy; suggesting that the procedure is robust to the non-normality of the innovation process.

V. Empirical Results

In this section we will investigate the time series properties of the real exchange rate for 22 industrial countries. PPP theory holds that there is an equilibrium level to which real exchange rates converge such that a unit of one currency should buy the same basket of goods in any country. By examining the time series properties of real exchange rates we can determine whether real exchange rates do converge in the long-run, even at a relatively slow speed of say 5 percent a month, and thus determine whether PPP is consistent with the data.

In this paper, the long run is taken to be the length required to reject the unit root hypothesis if a stationary alternative is true. Suppose that the speed of adjustment is 5 percent per month then the required length of time is 24 years.[5] Using only a single exchange rate one would not expect to reject the unit root model with a sample of less than 24 years. However, in a sample of 20 or so countries with at least 15 years of data we would expect approximately half of them to reject the unit root model if the speed of adjustment were as low as 5 percent per month.

The data used to estimate the near unit root model are monthly time series of the real exchange rate obtained from the International Financial Statistics (IFS) over the sample 1979:1 to 1995:6, which gives a total of 198 observations. Three definitions of the real exchange rate are considered: (i) the REER based on consumer prices (line *rec*) for which 22 industrial countries where selected,[6] (ii) the REER based on normalized unit labor costs (line *reu*) for which 17 industrial countries where selected,[7] and (iii) real

[5]The 10 percent critical value is -11·1. Thus if the speed of adjustment is 5 percent per month (or 54 percent per year) then the required time is $T = 11 \cdot 1/(1 - \cdot 54)$ which equals 24 years.

[6]The countries and IFS line numbers are: United States (111), Canada (156), Australia (193), Japan (158), New Zealand (196), Austria (122), Belgium (124), Denmark (128), Finland (172), France (132), Germany (134), Greece (174), Iceland (176), Ireland (178), Italy (136), Netherlands (138), Norway (142), Portugal (182), Spain (184), Sweden (144), Switerland (146), and United Kingdom (112). For a detailed explanation of how the REER is constructed see pages 60 and 61 in recent editions of the IFS.

[7]The countries are the same as above excluding Australia, New Zealand, Greece, Iceland, and Portugal.

bilateral exchange rates based on stock market prices for which 18 countries where selected.[8] All variables are transformed to natural logarithms.

Chart 4 shows the Japanese real exchange rate for each of the three definitions considered. The different measures of the real exchange rate are highly correlated showing common turning point and the general appreciation of the exchange rate through-out the 1980's, that would be typical of a process with a unit root.

We now describe the results of the near unit root estimation for the REER series which are reported in Tables 3 and 4. Table 3 contains the results for when a constant was used in the model while Table 4 contains the results for when a constant and a time trend was used in the model. In all cases except one (Norway), the 90% confidence interval of the autoregressive coefficient encompassed unity. In many cases the median unbiased point estimate indicated a moderately explosive model. The real effective exchange rate for Norway provided the lowest point estimate in both the model with a constant (·939) and the model with a time trend (·951).

In order to summarize the results of the near unit root estimation, a density estimator for the autoregressive parameter is computed using both the initial estimator $\widehat{\alpha}^{+}$ and the asymptotic median unbiased estimator $\widehat{\alpha}_{AMU}$. The procedure involves taking the estimated autoregressive parameter from the sample of industrial countries and forming a density plot.[9] The results of the density estimation are presented in Charts 5 through 8. Examining these charts we see that the estimated densities of the autoregressive parameter using the initial estimator $\widehat{\alpha}^{+}$ is heavily skewed with long left tails making the determination of whether the REER contains a unit root difficult. However, the estimated density using the median unbiased estimator $\widehat{\alpha}_{AMU}$ is virtually symmetric around a value slightly larger than unity. The density computed from the REER data based on labor costs has longer left tails than those based on consumer prices but there is still insufficient information to dismiss the unit root model. For example, if one expected the speed of adjustment of the real exchange rate to a long-run equilibrium to be five percent a month then we would have expected that about 11 of the countries (50 percent of the sample) to produce estimated autoregressive parameters (from the model with a constant) to be less than ·93 (the critical value for the test of a unit root), whereas no point estimate was less than this critical value and only one point estimate even came close.

[8]For ease of comparison the bilateral exchange rates (U.S. dollars per unit of national curriencies) was used in an index form on the basis of 1990=100 (line *ahx*). The real exchange rates were constructed as $x = e - p + p*$, where e is the log nominal exchange rate, p and p^* are the logarithms of U.S. price index and the foreign price index (line 62). The countries are the same as in case (i) excluding Greece, Iceland, and Portugal. Austria, France, and Italy have restricted samples due to the unavailabilty of current equity price data. The Austrian sample ends 1995:4; the French sample ends 1993:12; and the Italian sample ends 1995:5.

[9]In order to present the results in a clear manner the estimated densities have been smoothed using the kernel method with a Gaussian kernel of the form $w(x) = \frac{1}{\sqrt{2\pi}}e^{-(1/2)x^2}$. For details on density estimation see Silverman (1986).

Table 3. Median-Unbiased Estimation and 90 Percent Confidence Interval of α for Real Effective Exchange Rate Series

(Using a constant in the model)

	REER based on					
	Consumer Prices			Normalized Unit Labor Costs		
Country	α^+	α_{AMU}	$[\alpha_L, \alpha_U]$	α^+	α_{AMU}	$[\alpha_L, \alpha_U]$
United States	.989	1.007	[.992,1.021]	.993	1.008	[.995,1.022]
Canada	.996	1.009	[.997,1.022]	.919	.978	[.939,1.012]
Australia	.979	1.005	[.984,1.020]	...	...	[... , ...]
Japan	.995	1.009	[.996,1.022]	.998	1.009	[.999.1.022]
New Zealand	.929	.984	[.946,1.015]	...	...	[... , ...]
Austria	.995	1.009	[.996,1.022]	.983	1.006	[.987,1.021]
Belgium	.963	1.002	[.971,1.018]	.971	1.003	[.977,1.019]
Denmark	.968	1.003	[.975,1.019]	.995	1.009	[.996,1.022]
Finland	.979	1.005	[.984,1.020]	.988	1.007	[.991,1.021]
France	.957	1.000	[.967,1.018]	.934	.987	[.949,1.015]
Germany	.962	1.001	[.971,1.018]	1.004	1.010	[1.002,1.023]
Greece	.955	.999	[.964,1.018]	...	...	[... , ...]
Iceland	.918	.978	[.938,1.012]	...	...	[... , ...]
Ireland	.940	.990	[.953,1.016]	1.001	1.010	[1.000,1.022]
Italy	.987	1.007	[.990,1.021]	.991	1.008	[.994,1.021]
Netherlands	.941	.991	[.954,1.016]	.955	.999	[.964,1.018]
Norway	.951	.997	[.962,1.071]	.848	.939	[.891, .992]
Portugal	.988	1.007	[.991,1.021]	...	...	[... , ...]
Spain	.983	1.006	[.987,1.020]	.974	1.004	[.980,1.020]
Sweden	.968	1.003	[.975,1.019]	.972	1.004	[.979,1.019]
Switzerland	.969	1.003	[.976,1.019]	.966	1.002	[.974,1,019]
United Kingdom	.947	.995	[.959,1.017]	.933	.986	[.948,1.015]

Data source: International Financial Statistics.

Chart 5: Estimator densities for REER
(based on consumer prices)

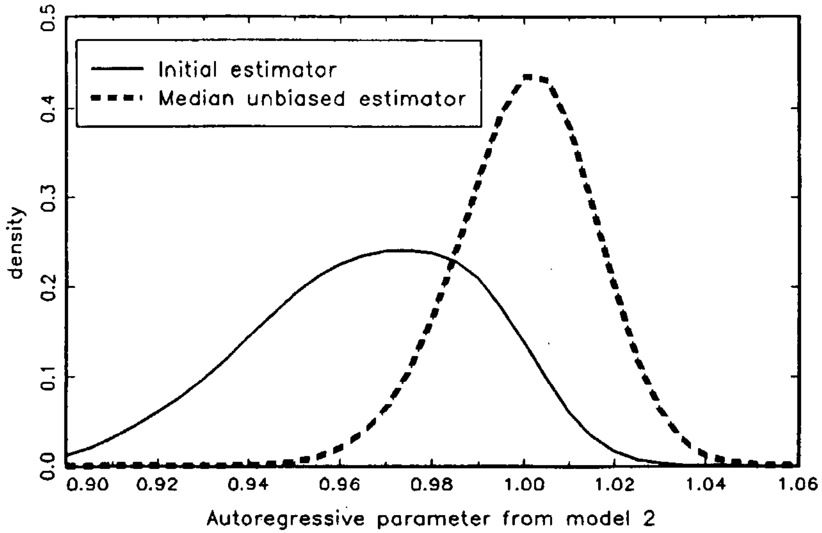

Chart 6: Estimator densities for REER
(based on consumer prices)

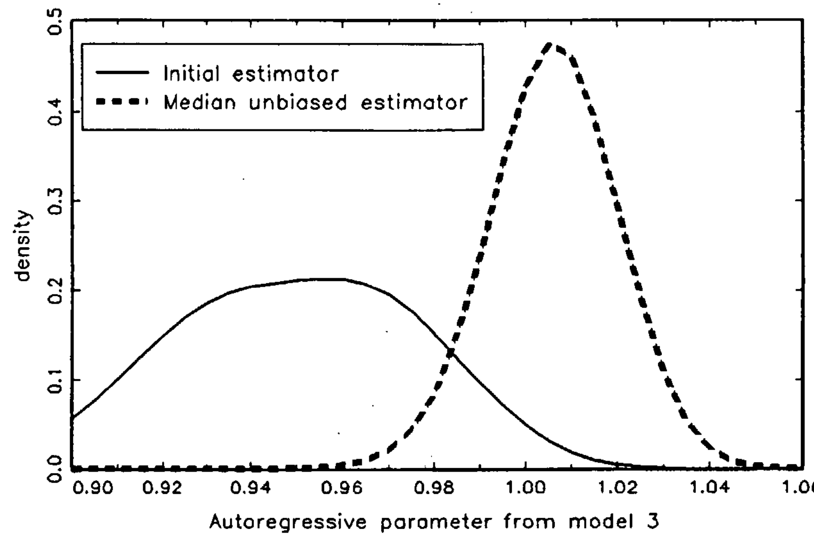

Chart 7: Estimator densities for REER
(based on unit labor costs)

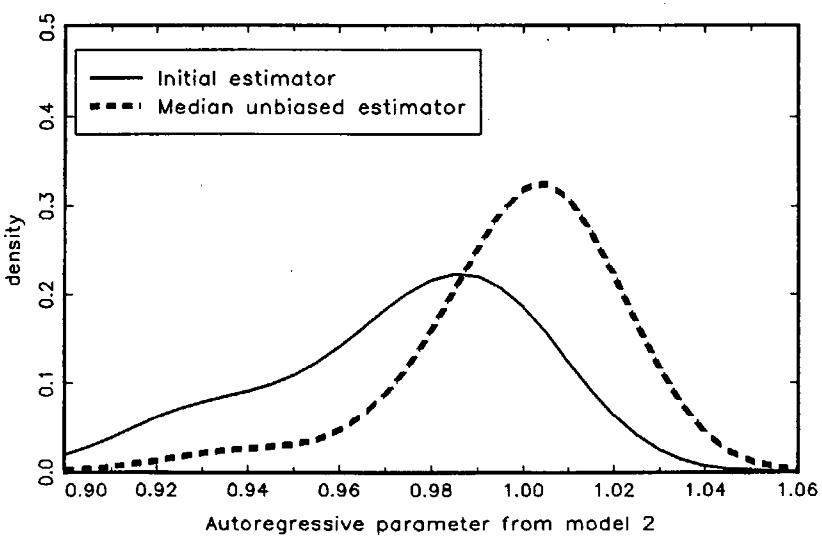

Chart 8: Estimator densities for REER
(based on unit labor costs)

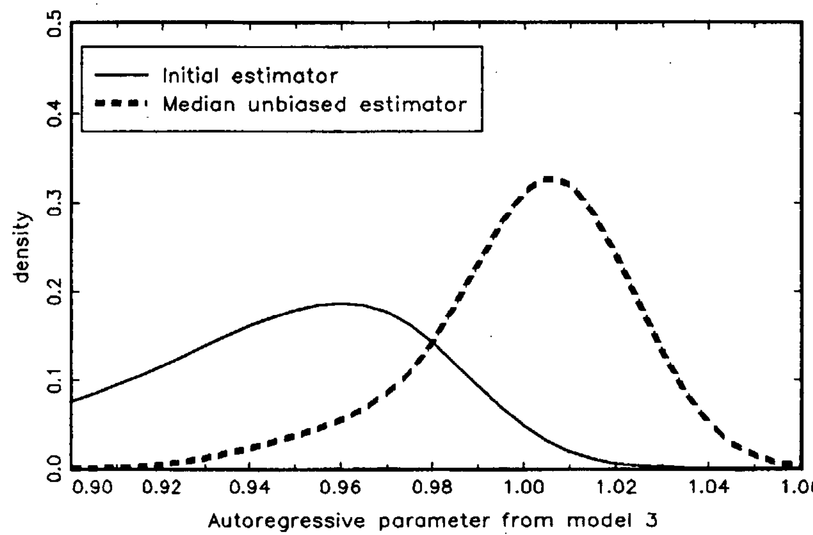

Table 4. Median-Unbiased Estimation and 90 Percent Confidence Interval of α for Real Effective Exchange Rate Series

(Using a constant and time trend in the model)

	REER based on					
	Consumer Prices			Normalized Unit Labor Costs		
Country	α^+	α_{AMU}	$[\alpha_L, \alpha_U]$	α^+	α_{AMU}	$[\alpha_L, \alpha_U]$
United States	.974	1.010	[1.000,1.024]	.969	1.010	[.995,1.024]
Canada	.987	1.013	[1.003,1.025]	.921	1.002	[.948,1.020]
Australia	.941	1.005	[.965,1.022]	...	...	[..., ...]
Japan	.932	1.004	[.957,1.021]	.932	1.003	[.957,1.021]
New Zealand	.926	1.003	[.952,1.020]	...	...	[..., ...]
Austria	.925	1.002	[.951,1.020]	.937	1.004	[.961,1.021]
Belgium	.968	1.009	[.992,1.024]	.977	1.011	[1.001,1.024]
Denmark	.940	1.005	[.964,1.021]	.941	1.005	[.965,1.022]
Finland	.979	1.011	[1.001,1.025]	.972	1.010	[.999,1.024]
France	.952	1.007	[.975,1.022]	.884	.973	[.922,1.017]
Germany	.963	1.009	[.984,1.023]	.946	1.006	[.969,1.022]
Greece	.957	1.008	[.980,1.023]	...	...	[..., ...]
Iceland	.903	.988	[.935,1.018]	...	...	[..., ...]
Ireland	.930	1.003	[.955,1.021]	.913	1.000	[.942,1.019]
Italy	.994	1.014	[1.005,1.027]	.986	1.013	[1.003,1.025]
Netherlands	.944	1.006	[.967,1.022]	.961	1.008	[.983,1.023]
Norway	.946	1.006	[.969,1.022]	.848	.951	[.898,1.012]
Portugal	.965	1.009	[.989,1.023]	...	...	[..., ...]
Spain	.975	1.011	[1.000,1.024]	.974	1.011	[1.000,1.024]
Sweden	.968	1.003	[.975,1.019]	.962	1.009	[.984,1.023]
Switzerland	.911	1.000	[.941,1.019]	.951	1.007	[.974,1,022]
United Kingdom	.920	1.001	[.947,1.020]	.900	.984	[.933,1.018]

Data source: International Financial Statistics.

We now describe the results for the of the near unit root estimation for real exchange rate series based on equity prices which are reported in Table 5. Evidence from the bilateral exchange rate series suggest the existence of a unit root. Japan and Spain even display explosive behavior with the lower 5 percent quantile being in excess of one. The explosive behavior in Japan in due to the sharp increases in assets prices in the late 1980's in contrast to the generally stable prices of goods and services.

Again to summarize the results of the near unit root estimation, a density estimator for the autoregressive parameter is computed. The result of the density estimation are presented in Charts 9 and 10. Examining these two charts we can observe that estimated densities of the autoregressive parameter using the initial estimator $\widehat{\alpha}^{+}$ on Models 2 and 3 are heavily skewed and the proportion of point estimators at or above unity is 0 percent using both Models 2 and 3. The density estimation based on the median unbiased estimator $\widehat{\alpha}_{AMU}$ is again symmetric at or slightly above unity with approximately 70 percent of Model 2 and 80 percent of Model 3 point estimates being at or above unity.

Overall the evidence against the unit root hypothesis is very weak and inconsistent with any stationary alternative that holds any economic meaning. To understand the difference between the unit root model and a stationary AR model with a high degree of correlation, simulated density estimates for a unit root process and a stationary AR process were generated using standard normal errors, a time length of 200 observations and 22 "country" observations. The estimated densities from the simulated data are shown in Charts 11 and 12. In the unit root case, the mode and the median of the simulated density estimate (based on a median unbiased estimator) lies at unity but the density is more dispersed and skewed than the density produced using actual real exchange rate data. In the stationary case, the simulated density estimate is even more dispersed but symmetric around its true value of ·95, as would be predicted by standard asymptotic theory for stationary time series.

VI. Conclusion

This paper has re-examined whether PPP holds in the long-run by investigating the time series properties of the real exchange rate for 22 industrial countries. PPP theory holds that there is an equilibrium level to which real exchange rates converge such that a unit of one currency should buy the same basket of goods in any country. By examining the time series properties of real exchange rates we can determine whether real exchange rates do converge in the long-run, even at a relatively slow speed of say 5 percent a month.

Since only Norway's real exchange rate produced a 90 percent confidence interval that did not include unity we can conclude that overall the unit root model gives the best

Chart 9: Estimator densities
(based on bilateral rates)

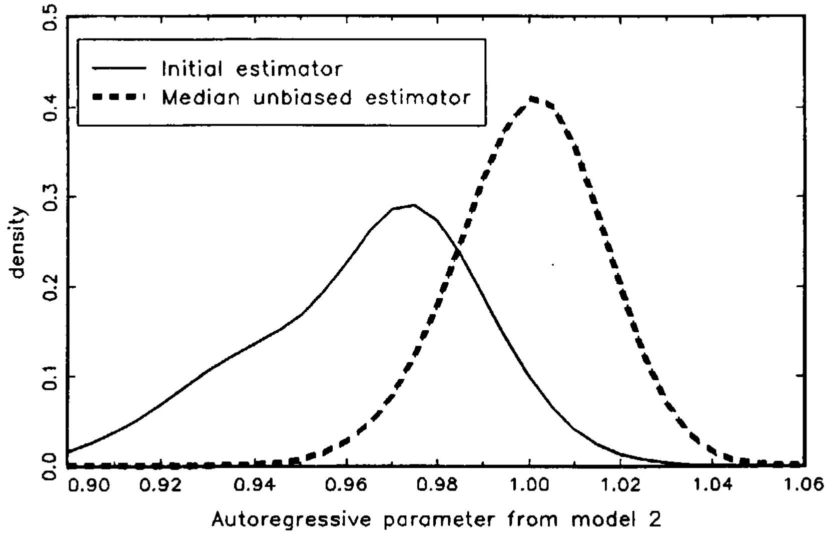

Chart 10: Estimator densities
(based on bilateral rates)

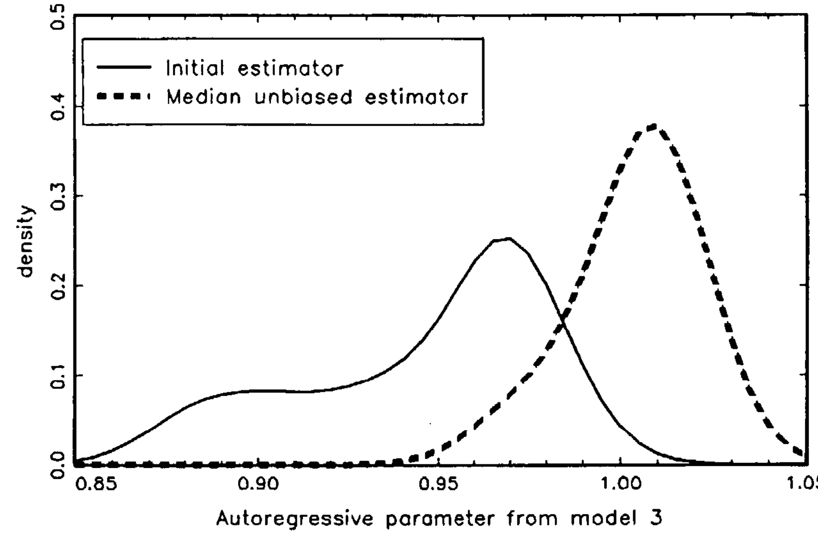

Chart 11: Estimator densities
(for unit root model)

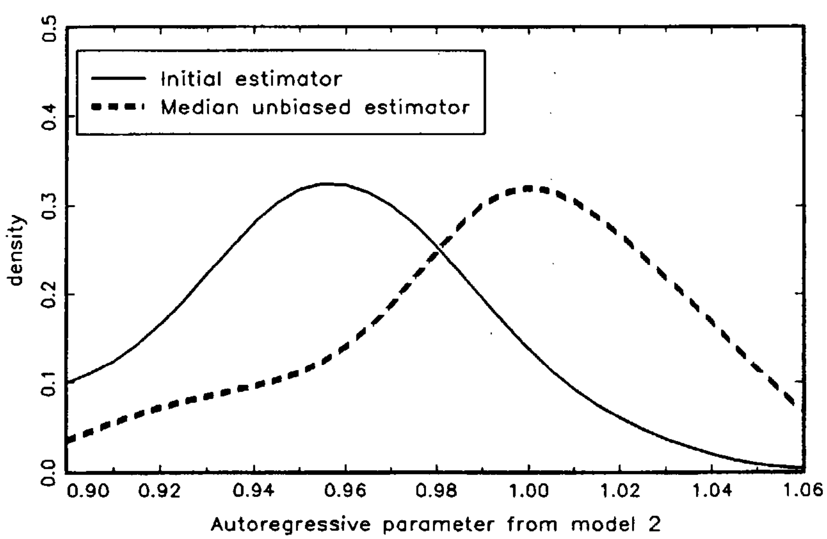

Chart 12: Estimator densities
(for AR(1) model with coefficient=0.95)

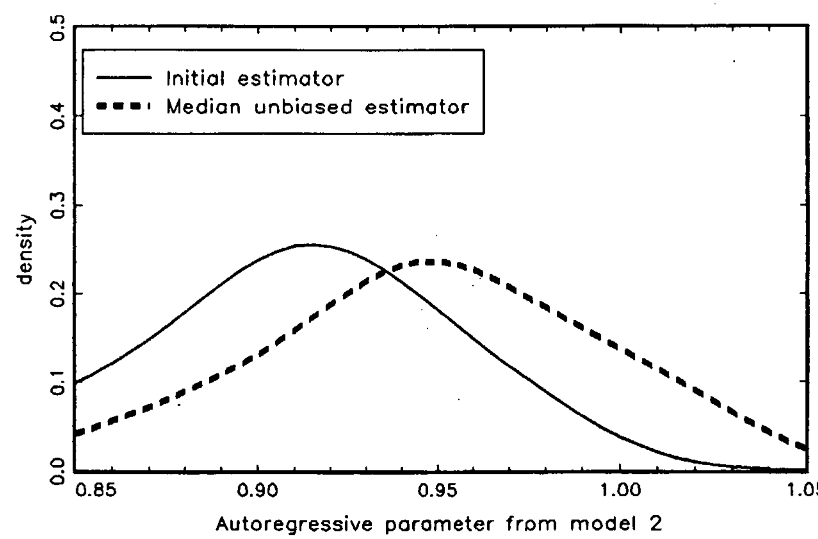

Table 5. Median-Unbiased Estimation and 90 Percent Confidence Interval of α for Equity Price based Real Exchange Rate Series

Country[1]	constant in model			time trend in model		
	α^+	α_{AMU}	$[\alpha_L, \alpha_U]$	α^+	α_{AMU}	$[\alpha_L, \alpha_U]$
Canada	.999	1.010	[.999,1.022]	.881	.972	[.920,1.017]
Australia	.968	1.003	[.975,1.019]	.905	.991	[.936,1.019]
Japan	.984	1.006	[.988,1.021]	.982	1.012	[1.002,1.025]
New Zealand	.963	1.002	[.971,1.018]	.953	1.007	[.976,1.022]
Austria	.976	1.005	[.982,1.020]	.968	1.010	[.992,1.024]
Belgium	.969	1.003	[.976,1.019]	.969	1.010	[.993,1.024]
Denmark	.940	.990	[.953,1.016]	.924	1.021	[.950,1.020]
Finland	.981	1.006	[.986,1.020]	.973	1.010	[.999,1.024]
France	.977	1.005	[.981,1.022]	.966	1.010	[.988,1.026]
Germany	.938	.989	[.952,1.016]	.939	1.005	[.963,1.021]
Ireland	.970	1.003	[.977,1.019]	.969	1.010	[.995,1.024]
Italy	.968	1.003	[.975,1.019]	.969	1.010	[.993,1.024]
Netherlands	.917	.977	[.937,1.012]	.904	.989	[.936,1.019]
Norway	.944	.993	[.956,1.016]	.884	.973	[.922,1.017]
Spain	.986	1.007	[.989,1.021]	.984	1.012	[1.003,1.025]
Sweden	.979	1.005	[.984,1.020]	.969	1.010	[.993,1.024]
Switzerland	.968	1.003	[.975,1.019]	.959	1.008	[.981,1.023]
United Kingdom	.935	.988	[.950,1.016]	.935	1.004	[.960,1.021]

Data source: International Financial Statistics.

[1]The bilateral exchange rates (U.S. dollars per unit of national curriencies) were used in an index form on the basis of 1990=100. The real exchange rates were constructed as $x = e - p + p^*$, where e is the log nominal exchange rate, p and p* are the logarithms of U.S. price index and the foreign price index.

description of real exchange rates in industrial countries. This conclusion was reached using econometric methods that correct for the bias present in typical AR model estimators (such as least squares and the Phillips-Perron estimator) and that combine country information on the time series properties of real exchange rates in a graphical manner.

The finding that real exchange rate movements do not conform to PPP has several implications. First, the purchasing power of a given income in one country and currency cannot be easily compared. When actual exchange rates are used to make such comparisons the real income of countries will be biased. Second, deviations from PPP imply informational inefficiency in the sense that rational investors, using available information, could make excess profits by borrowing in one country, and buying and holding stocks in a another. Third, persistent deviations from PPP imply persistent swings in a country's competitiveness. These deviations in competitiveness can yield large deviations in a country's external balance, output, and employment.

Appendix: The Econometric Assumptions

The following assumption is sufficient for a *functional central limit theorem* to be applied to partial sums of u_t and u_t^2 and has been employed in many papers for such a purpose. For examples and further discussion see Phillips (1987a, b), Phillips and Perron (1988) and Perron (1988).

Assumption 2 *(a)* $E(u_t) = 0$ *for all* t;
(b) $\sup_t E|u_t|^{\beta+\epsilon} < \infty$ *for some* $\beta > 2$ *and* $\epsilon > 0$;
(c) $\omega^2 = \lim_{T\to\infty} E(T^{-1}S_T^2)$ *exists and* $\omega^2 > 0$, *where* $S_t = \sum_{k=1}^{t} u_k$;
(d) $\{u_t\}$ *is strong mixing with mixing coefficients* α_m *that satisfy* $\sum_{m=1}^{\infty} \alpha_m^{1-2/\beta} < \infty$.

If $\{u_t\}$ is weakly stationary with spectral density $f_u(\lambda)$, then the long-run variance of u_t is

$$\omega^2 = \sum_{j=-\infty}^{\infty} E(u_0 u_j) = 2\pi \; f_u(0) > 0. \tag{9}$$

The standard variance of u_t is

$$\sigma_u^2 = \lim_{T\to\infty} T^{-1} \sum_{t=1}^{T} E(u_t^2). \tag{10}$$

For the sequence of partial sums $\{S_t : S_t = \sum_{k=0}^{t} u_k\}$, we construct

$$\begin{aligned} X_T(r) &= T^{-1/2}\omega^{-1} S_{[Tr]} \\ &= T^{-1/2}\omega^{-1} S_{j-1}, \quad \tfrac{j-1}{T} \le r < \tfrac{j}{T} \quad (j = 1, ..., T) \end{aligned} \tag{11}$$

where $[Tr]$ denotes the integer part of Tr. We also denote by $D[0,1]$ the space of *cadlag* functions,[10] endowed with the uniform metric $\|f - g\| = \sup_r |f(r) - g(r)|$ for any $f, g \in D[0,1]$. $X_T(r)$ is a random element in the function space $D[0,1]$ and obeys a *functional central limit theorem* or *invariance principle* often expressed by the notation $X_T(r) \Rightarrow W(r)$. The symbol '$\Rightarrow$' signifies weak convergence of the associated probability measures as $T \to \infty$, and the limit process $W(r)$ is standard Brownian motion on $C[0,1]$.[11] When c is close to zero and thus α is local-to-unity, we can use a *functional central limit theorem* in which $X_T(r)$ converges weakly to a diffusion process $J_c(r)$ as $T \to \infty$, where

$$J_c(r) = \int_0^r e^{(r-s)c} dW(s). \tag{12}$$

We now give the assumption on the types of kernels $w(.)$ and bandwidth parameter b required for estimation.

[10] *cadlag* functions are real-valued functions on [0,1] that are right continuous at each point of [0,1) with left limits existing at each point of (0,1]. For a detailed discussion of $D[0,1]$ see Pollard(1984).
[11] $C[0,1]$ is the space of real-valued continuous functions on [0,1].

Assumption 3 : *(a) The kernel function $w(.): \mathbf{R} \to [-1,1]$ satisfies the following conditions: (i) $w(0) = 1$, (ii) $w(x) = w(-x)$ $\forall x \in \mathbf{R}$, (iii) $\int_{-\infty}^{\infty} |w(x)|dx < \infty$, and (iv) $w(.)$ is continuous at zero and at all but a finite number of other points,*

(b) The bandwidth parameter satisfies the condition $b \to \infty$ as $T \to \infty$ such that $b = o(T^{1/2})$.

Assumption 3 describes a class of kernels that can be considered. This class includes the Bartlett, Parzen, Tukey-Hanning and quadratic spectral kernels. Examples of suitable kernels can be found in Priestley (1981) and Andrews (1991). Assumptions 2 and 3 ensure the consistency of the estimator s_{Tb}^2 (see proof of Result 1). Andrews (1991) provides more primitive conditions for consistency of s_{Tb}^2 and also shows that the conditions on the bandwidth parameter could be relaxed so that $b = o(T)$.

Andrews (1991) shows that under more stringent primitive conditions than in Assumption 3, s_{Tb}^2 is consistent when the bandwidth parameter b is chosen by an automatic "plug-in" selection procedure. The assumption required when b is chosen by the data dependent method of Andrews (1991) is given below.

Assumption 4 : *(a) The kernel function $w(.): \mathbf{R} \to [-1,1]$ satisfies the conditions of Assumptions 3(a) and (i) $w(x) \leq C_1|x|^{-B}$ for some $B > 1 + 1/q$ and some $C_1 < \infty$, where $q \in (0,\infty)$ such that $w_q \in (0,\infty)$, and (ii) $|w(x) - w(y)| \leq C_2|x-y|$ $\forall x, y \in \mathbf{R}$ for some constant $C_2 < \infty$.*

(b) $\hat{b} = \hat{\phi}T^{1/(2q+1)} - 1$, where $\hat{\phi}$ satisfies $\hat{\phi} = O_p(1)$ and $1/\hat{\phi} = O_p(1)$.

We now recall a useful Lemma from Phillips (1987*b*).

Lemma 1 : *Let Assumptions (1), (2), and (3) hold. Then as $T \to \infty$,*

(a) $T^{-1/2}Y_{[T\tau]} \Rightarrow \omega\ J_c(r)$;

(b) $T^{-3/2}\sum_{t=1}^{T} Y_t \Rightarrow \omega\ \int_0^1 J_c(r)dr$;

(c) $T^{-2}\sum_{t=1}^{T} Y_t^2 \Rightarrow \omega^2 \int_0^1 J_c(r)^2 dr$;

(d) $T^{-1}\sum_{t=2}^{T} Y_{t-1}u_t \Rightarrow \omega^2 \int_0^1 J_c(r)dW(r) + (\omega^2 - \sigma_u^2)/2$, where $\sigma_u^2 = \lim_{T\to\infty} T^{-1}\sum_{t=1}^{T} E(u_t^2)$. Joint weak convergence of (a)-(d) also applies.

We now recall a useful result, originally reported in Phillips (1987*b*) and Phillips and Perron (1988), on local-to-unity asymptotic distributions. The result delivers a distribution which can be used to derive a median-unbiased estimator. Phillips (1987*b*) provides results for Model 1 and Phillips and Perron (1988) extend these results to Models 2 and 3. In this paper we make extensive use of these results and for completeness the proofs are also provided.

Result 1 : *Let Assumptions (1), (2) and (3) hold. Then as $T \to \infty$,*

$$T(\hat{\alpha}_i^+ - 1) \Rightarrow c + \frac{\int_0^1 J_c^i(r)dW(r)}{\int_0^1 \{J_c^i(r)\}^2 dr}, \qquad i = 1,2,3;$$

where $\hat{\alpha}_i^+$ is the estimator for Model i, $J_c^1(r) = J_c(r)$, $J_c^2(r) = J_c(r) - \int_0^1 J_c(r)dr$ and $J_c^3(r) = J_c(r) - \int_0^1 (4-6r)J_c(r)dr - r\int_0^1 (12r-6)J_c(r)dr$.

PROOF OF RESULT 1: Here we prove the results for Model 1 only since the results for Model 2 and 3 follow analogously (see also Phillips and Perron (1988, Theorem 3, p.342)). We claim that

$$s_u^2 \xrightarrow{p} \sigma_u^2 \tag{13}$$

and

$$s_{Tb}^2 \xrightarrow{p} \omega^2. \tag{14}$$

We note that in the case $i = 1$

$$\begin{aligned} T(\hat{\alpha}_1^+ - \alpha) &= \left(T^{-2}Y_{-1}'Y_{-1}\right)^{-1}\left(T^{-1}Y_{-1}'Y - \hat{\lambda}_1 - \alpha/T\right) \\ &= \left(T^{-2}Y_{-1}'Y_{-1}\right)^{-1}\left(T^{-1}Y_{-1}'U - (s_{Tb}^2 - s_u^2)/2\right) \\ &\Rightarrow \int J_c(r)dW(r) \int J_c(r)^2 dr. \end{aligned}$$

Weak convergence follows by direct application of the continuous mapping theorem and Lemma 1, together with (13) and (14).

It remains to show (13) and (14). Under Assumption 2, (13) follows by applying a strong law of large numbers for dependent heterogeneously distributed sequences. In particular, see Theorem 2.10 of McLeish (1975). To show (14) define

$$\sigma_{Tb}^2 = T^{-1}\sum_{t=1}^{T} E(u_t^2) + 2\,T^{-1}\sum_{\tau=1}^{b} w\left(\frac{\tau}{b+1}\right)\sum_{t=\tau+1}^{T} E(u_t u_{t-\tau}). \tag{15}$$

It suffices to show that $s_{Tb}^2 - \sigma_{Tb}^2 \xrightarrow{p} 0$ and that $\sigma_{Tb}^2 - \omega^2 \xrightarrow{p} 0$. Under Assumptions 2 and 3, $s_{Tb}^2 - \sigma_{Tb}^2 \xrightarrow{p} 0$ and $\sigma_{Tb}^2 - \omega^2 \xrightarrow{p} 0$ follow from Theorem 1(a) of Andrews (1991). Hence $s_{Tb}^2 \xrightarrow{p} \omega^2$ completing the proof. □

We now provided an invariance result for the estimators defined in (2). Similar invariance results have been shown by Andrews (1993), DeJong *et al.* (1992), and Dickey and Fuller (1979). The invariance property of $\hat{\alpha}_i^+$ shown in Result 2 is extremely useful in simulating the bias correction needed for estimators of α.

Result 2 : *The distribution of $\hat{\alpha}_1^+$ is invariant with respect to σ^2, the distribution of $\hat{\alpha}_2^+$ is invariant with respect to (σ^2, μ), and the distribution of $\hat{\alpha}_3^+$ is invariant with respect to (σ^2, μ, β).*

PROOF OF RESULT 2: Consider Model 3. The first step is to show the distribution of $\hat{\alpha}_3^+$ is invariant with respect to (μ, β). Define $R = Y - X_3(X_3'X_3)^{-1}X_3'Y$ and $R_{-1} = Y_{-1} - X_3(X_3'X_3)^{-1}X_3'Y_{-1}$. Note that R and R_{-1} are in invariant with respect to (μ, β) and thus $\hat{\alpha}_{LS3} = [R_{-1}'R_{-1}]^{-1}R_{-1}'R$ is invariant. Hence the distribution of $\hat{\alpha}_{LS}$ is invariant with respect to (μ, β). Since $\hat{\alpha}_3^+ = \hat{\alpha}_{LS3} + [Y_{-1}'(I - P_3)Y_{-1}]^{-1}T\hat{\lambda}_3 =$ $\hat{\alpha}_{LS3} = [R_{-1}'R_{-1}]^{-1}T\hat{\lambda}_3$, it suffices to show that $\hat{\lambda}_3$ is invariant with respect to (μ, β). Recall $\hat{\lambda}_3 = 2\,T^{-1}\sum_{\tau=1}^{b} w\left(\frac{\tau}{b+1}\right)\sum_{t=\tau+1}^{T}(u_{LS3,t}u_{LS3,t-\tau})$ and $\hat{u}_{LS3} = R - \hat{\alpha}_{LS3}R_{-1}$. Since $\hat{u}_{LS3}$ is invariant with respect to (μ, β) then so is $\hat{\lambda}_3$. The second step is to show that the distribution of $\hat{\alpha}_3^+$ is invariant with respect to σ_u^2. Divide σ_u^2 by a nonzero constant d to obtain

$$\begin{aligned} Y_t/d &= \mu/d + (\beta/d)t + Y_t^*/d \\ Y_t^*/d &= \alpha(Y_{t-1}^*/d) + u_t/d. \end{aligned} \tag{16}$$

Let estimates for (16) be denoted by the superscript d. Notice that

$$\begin{aligned} \hat{\alpha}_{LS3}^d &= [(R_{-1}/d)'(R_{-1}/d)]^{-1}(R_{-1}/d)'(R/d) \\ &= \hat{\alpha}_{LS3} \\ \hat{u}_{LS3}^d &= (R/d) - \hat{\alpha}_{LS}(R_{-1}/d) \\ &= \hat{u}_{LS3}/d \\ \hat{\lambda}_3^d &= 2\,T^{-1}\textstyle\sum_{\tau=1}^{b} w\left(\frac{\tau}{b+1}\right)\sum_{t=\tau+1}^{T}(u_{LS3,t}^d u_{LS3,t-\tau}^d) \\ &= \hat{\lambda}_3/d^2. \end{aligned}$$

Hence

$$\begin{aligned} \hat{\alpha}_{LS3}^{+d} &= \hat{\alpha}_{LS3}^d + [(R_{-1}/d)'(R_{-1}/d)]^{-1}T\hat{\lambda}_3^d \\ &= \hat{\alpha}_{LS3} + [(R_{-1}'R_{-1})/d^2]^{-1}T\hat{\lambda}_3/d^2 \\ &= \hat{\alpha}_{LS3}^+. \end{aligned}$$

The result follows since $\hat{\alpha}_{LS3}^+$ is invariant with respect to $(\mu/d, \beta/d)$. The proofs for Models 1 and 2 are analogous. □

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