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Long Memory Processes and Chronic Inflation: Detecting Homogeneous Components in a Linear Rational Expectation Model

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Abstract

This paper is an empirical study of the links between monetary variables and inflation based on Cagan's equation and its rational expectations solution, when the forcing variable is a fractionally integrated process. As demonstrated by Hamilton and Whiteman, the existence of bubbles and other extraneous influences can be detected only by verifying the difference in the order of integration between the monetary base and the price level series. This paper shows that a fractionally differenced model overcomes Evans' critique of this test and that chronic inflation is essentially a monetary phenomenon caused by fiscal imbalance.

JEL Classification No.

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Summary

Pathological economic phenomena like hyperinflation or chronic inflation have been often attributed to nonfundamental influences, which implies that they are difficult to control by conventional policy instruments. In the literature, such influences have been given different names--for example, bubbles, sunspots, or extraneous effects--but essentially they all represent the homogenous component of the solution to an expectational equation. Hamilton and Whiteman (1985), extending the results by Flood and Garber (1980), have shown that the presence of nonfundamental influences can only be verified by analyzing the difference between the dynamics of the narrow money supply and the price level series, as reflected in their respective orders of integration. If the former is exogenous with respect to the latter, a nonfundamental influence can be assumed to exist.

Evans (1990) criticized this approach, asserting that conventional tests fail to reject the hypothesis that a particular exploding bubble process is nonstationary. Accordingly, he concluded that the Hamilton and Whitman approach has limited practical application.

This paper illustrates the reasons for Evans' results and explains how his critique can be overcome by using fractionally integrated processes, which allow greater flexibility in modelling long-cycle components of nonstationary time series by extending the standard time-series methodology to noninteger orders of integration.

The empirical part of the paper focuses on the estimation of the fractional orders of integration of money supply and price level in six countries using an exact maximum-likelihood method devised by Sowell (1990). The objective was to test the hypothesis that high inflation is caused by self-fulfilling expectations, as reflected by a homogeneous term in the solution to the expectational equation. For all six countries, the presence of an homogeneous component is rejected. However, in a few cases there is a slight difference in the order of integration, which is attributed to protection from inflation accorded to particular groups, for example in the form of wage indexation.

I. Introduction

Recent studies on inflation have focused on Cagan's equation and its rational expectations solution, which relates general price index movements to the money supply. Despite its intrinsic simplicity, this equation can generate a wide range of dynamics, including both stable and unstable laws of motion, self-fulfilling speculative price paths, "bubbles," or completely extraneous influences, "sunspots."

The idea that increases in consumer price index may not reflect movements in fundamentals is hardly new. The idea, *in nuce*, was already contained in Cagan (1956) and the problem of the non-uniqueness of equilibrium in a dynamic monetary economy has surfaced since Sargent and Wallace (1973b). Brock (1975) undertook a comprehensive treatment of stable and unstable solutions in a framework of intertemporal optimization with real money balances in the utility function. 1/ However it was the seminal article by Flood and Garber (1980) 2/ that renewed the interest in Cagan's model and made the notion of rational bubble popular, by showing how self-fulfilling expectations might arise in a rational expectation context derived as a particular case of Brock (1975). In linear dynamic models bubbles and other nonfundamental influences are represented by the homogeneous part of the solution to a linear difference equation and logically are not related to the way expectations are formed. What expectations affect is the persistence of inflation: adaptive or backward-looking expectations would delay the effect of any change in policy as the agents' reactions lag. 3/ In other words bubbles and persistence are conceptually distinct phenomena.

This paper will focus mainly on an empirical test of the presence of bubbles and sunspots based on the study of the solution to Cagan's equation by Hamilton and Whiteman (1985); generalizing the results by Burmeister, Flood and Garber (1980), they showed that bubbles, sunspots and related processes, are observationally equivalent to "fundamental" equilibria, once a fairly general dynamic specification for the driving variables has been postulated. Hence they concluded that the only falsifiable hypothesis implied by self-fulfilling expectations concerns the difference in the order of integration of the relevant variables. In the presence of bubbles money supply has a lower order of integration than the price level or, stated differently, changes in money supply are followed by larger movements in the price level. 4/

1/ A critical review of this literature is contained in Gray (1984).

2/ Kingston (1982) proved that Cagan's equation can be derived as a special case of the general equilibrium model in Brock (1975).

3/ See Sargent (1986) introduction to Chapter 3, "The end of four big inflations," for a lucid and concise treatment of the dichotomy between adaptive versus rational expectations.

4/ Even after Hamilton and Whiteman (1985) was published the fundamental importance of this falsifiable hypothesis is not always perceived in the literature. For example, Casella (1989), in her analysis of the post World War I German hyperinflation, takes the second difference of the data, therefore assuming, without testing, that the series have the same order of nonstationarity.

Evans (1991) criticized Hamilton and Whiteman (1985), by showing in a Monte Carlo study, that a specific form of bubble process cannot be detected by conventional unit root tests.

This paper argues that Evans' critique can be overcome by exploiting the properties of a recently developed time series model, the so called Autoregressive Fractionally Integrated Moving Average (ARFIMA). 1/ These fractionally integrated processes, generalize the treatment of standard integrated processes by considering noninteger order of integration, thereby providing an accurate representation of time series with slowly deçaying autocovariance structures (also called long memory structures). The advantages of ARFIMA models in testing for the presence of bubbles can be synthesized in three points.

1. A greater degree of diagnostic precision than the standard stationarity tests (see also Diebold and Rudenbush (1988)) because, unlike ARIMA, ARFIMA do not place any restrictions on the long run characteristics of the series.

2. Separate analysis of the short-term and the long-term dynamics of a variable, which is of extreme importance in the study of inflation. In fact, the long-term dynamics account for shifts in fiscal regimes, while short-term effects are the results of what Sargent (1986) calls limited actions, i.e., policy measures which do not attack the root of inflation.

3. Extreme generality in the sense that, unlike other methods, ARFIMA-based tests do not require to specify the particular form of the bubble. With regard to Evans' critique, they can differentiate rapidly collapsing bubbles from noises, through a careful diagnosis of the long memory characteristics of the autocovariance function.

Nonfundamental influences have been long debated in economics and have been brought up in several circumstances to explain anomalies in speculative markets or macroeconomic data. For instance, bubbles may arise when the current value of an asset is determined (at least in part) by the expected rate of market price change. The mere self-fulfilling assessment of a future change can drive the current value to a level unwarranted by economic fundamentals.

The case of inflation is somehow analogous: inflation reflects the present value of expected future government deficits. Agents (firms and workers) base their economic decisions on these expectations, so even a temporary misperception can gain momentum and trigger a self-sustaining process of hyperinflation. For example, fearing a sudden loss of purchasing power, consumers would be induced to dispose of money by purchasing goods,

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1/ Two other ways to overcome this problem, were proposed by Hall and Sola (1993) and Blackburn and Sola (1993), who resorted to the Markov regime-switching model by Hamilton (1988, 1990).

thereby provoking further price increases; analogously trade unions in collective bargaining would push for higher wages, which would be transferred on prices.

The existence of bubbles or sunspots has far-reaching consequences for economic policy. In particular, if inflation expectations are self-sustaining or depend on causes beyond economic rationale, the price level will not respond to conventional monetary and fiscal measures. The cost of stabilization achieved through monetary and fiscal discipline, hence, will be extremely high as its pace will be very slow.

The nature of inflationary phenomena varies with their intensity and duration. Cagan (1956) defined hyperinflation an increase in the price level in excess of 50 percent per month. In this paper attention is paid to cases which is more appropriate to describe as chronic high inflation. Hyperinflation invariably does not last too long and likely agents anticipate that somehow it must come to an end. Chronic high inflation on the contrary protracts for a long span of time, so agents try to adjust to it and possibly to find some form of protection. Moreover it is less likely to be stopped by a sudden regime shift and therefore agents' decisions are prompted (in absence of bubbles) by the dynamics of the driving variable.

The thorough treatment of the topics summarized in this introduction is organized in five additional sections. Section II provides a brief outline of fractionally integrated processes. In Section III, a solution to Cagan's equation is formulated on the assumption that the driving variable, in this case money supply, is fractionally integrated. Section IV discusses Evans' critique, its relevance, and the reasons why ARFIMA models offer an appropriate response. The results of the empirical analysis for six countries (Bolivia, Former Socialist Federal Republic of Yugoslavia, Brazil, Argentina, Peru, and Chile) are given in Section V. Section VI concludes with a discussion on the implications for economic policy.

II. Fractionally Integrated Processes

Classical time series analysis is based on an extremely simplified view: long-run behavior and impulse response are determined by the degree of integration of a variable, which is assumed to be an integer number. If a variable X_t is integrated of degree zero, in symbols $X_t \sim I(0)$, its unconditional variance is finite and an innovation does not have a lasting effect, since its autocorrelation decays at an exponential rate. If $X_t \sim I(1)$, its variance is infinite, hence an innovation has a permanent effect since X_t is the sum of all previous innovations and its autocorrelation approaches unity. 1/ For variables integrated of degree

1/ Most of the research devoted to the treatment of nonstationary economic series has focused on cointegration. Two variables integrated of degree d (d being an integer) are said to be cointegrated if there exists a linear combination which is integrated of degree d-1.

two or three or more, the same reasoning applies to their first, second, ..., n-th differences.

However, a wider range of dynamics can be analyzed by considering fractional (noninteger) orders of integration, in particular, long-memory processes (see McLeod and Hipel (1978)) whose correlation decays at a geometric rate. 1/

ARFIMA processes can be thought of as a natural extension of the standard time-series models (Granger and Joyeaux (1980) and Hosking (1981)). A random walk X_t, i.e., an ARIMA(0,1,0), is the d-difference of a white noise $\epsilon(t) \sim N(0,\sigma^2)$, where $d = 1$. In symbols

$$\nabla X_t = (1-L)\, X_t = \epsilon_t$$

where L is the lag operator, i.e., $LX_t = X_{t-1}$. Similarly an ARFIMA(0,d,0) process is a fractionally differenced (Gaussian) noise, defined as the d-th difference of a white noise, with $d \in (-0.5, 0.5)$:

$$\nabla^d X_t = (1-L)^d\, X_t = \epsilon_t \tag{2.1}$$

In other words, the ARFIMA(0,d,0) process is the stationary solution $\{X_t,\ t \in (-\infty,\infty)\}$ to the above difference equation. For a proof of the uniqueness of the solution when $-0.5 < d < 0.5$, see Brockwell and Davis (1987).

1/ As observed by Granger (1966), most economic variables have a "typical spectral shape" dominated by low frequency components. Under these circumstances first-differencing the series leads to loss of useful information on the long-cycle properties of the data.

The difference operator $(1-L)^d$ can be obtained by a binomial expansion 1/

$$(1-L)^d = \sum_{k=0}^{\infty}\binom{d}{k}(-L)^k = 1-dL-\frac{1}{2}d(1-d)L^2-\frac{1}{6}d(1-d)(2-d)L^3-\ldots, \tag{2.2}$$

where $d > -1$. Alternatively, the operator $(1 - L)^d$ for any real number $d > -1$ can be expressed as

$$(1-L)^d = \sum_{j=0}^{\infty}\frac{\Gamma(-d+j)}{\Gamma(-d)\Gamma(j+1)}L^j, \tag{2.3}$$

where Γ is the gamma function or generalized factorial.

The fractionally differenced noise is the building block of a larger class of processes, the ARFIMA(p,d,q), defined as the d-th fractional difference of an ARMA(p,q), i.e.,

$$\phi(L)\ (1-1)^d X_t = \theta(L)\epsilon_t \tag{2.4}$$

where p and q are the order of an autoregressive component and a moving average component, respectively, so that $\phi(L)$ and $\theta(L)$ are polynomials in the lag operator L, of order p and q, and $-0.5 < d < 0.5$. As in the case of the ARFIMA(0,d,0), $\{X_t\}$ can be interpreted as the unique stationary solution to (2.4), or, if one prefers, as an ARMA process driven by a fractionally integrated noise.

The usual definitions of causality and invertibility typical of ARMA processes apply to ARFIMA: $\{X_t\}$ is causal if and only if $\phi(z) \neq 0$ for $|z| \leq 1$ and is invertible if and only if $\theta(z) \neq 0$ for $|z| \leq 1$.

1/ The binomial expansion of $(a+b)^n$ for noninteger n is an infinite series defined as

$$(a+b)^n = \binom{n}{0}a^n+\binom{n}{1}a^{n-1}b+\binom{n}{2}a^{n-2}b^2+\ldots\ldots = \sum_{k=0}^{\infty}\binom{n}{k}a^{n-k}b^k$$

If $a = 1$ and $b < 0$ the above expression becomes

$$(1-b)^n = 1+\frac{n}{1!}*(-b)^0+\frac{n(n-1)}{2!}*(-b^2)+\frac{n(n-1)(n-2)}{3!}*(-b^3)+\ldots\ldots\ldots$$

which is essentially 2.1 if we set $b = L$.

It is essential here to point out that when d > |0.5|, the process (X_t) is nonstationary. In other words, -0.5 and 0.5 represent the borderline between stationarity and nonstationarity.

This parametrization of a time series as an ARFIMA process has two main advantages. On the one hand, it includes the standard ARIMA models as a particular case, thereby providing a direct and intuitive link to standard time series literature. On the other hand, it allows a more accurate analysis of long-run dynamics.

Specifically in classic time series models, the AR or MA parameters cannot describe the long-run properties of data separately from the shorter-run dynamics. In contrast the parameter d of an ARFIMA process describes the long-run properties, while the ARMA parameters account for the short-run behavior. As described in more detail below, the parameter d is related to the low frequency component of the spectral density, and therefore, the degree of stationarity is directly linked to the degree of persistence of a series.

An ARFIMA(p,d,q), has a long term behavior which in the limit approaches that of an ARFIMA(0,d,0) with the same parameter d, since the autoregressive and moving average parameters have only a slight and decreasing influence on distant observations. Indicating by k the time lag between observations, the autocorrelation function of an ARFIMA(0,d,0) is

$$R(k) \approx k^{2d-1}\,\Gamma(1-d)/\Gamma(d) \qquad \text{as } k \to \infty$$

while for an ARFIMA(p,d,q), it is

$$R(k) \approx Ck^{2d-1} \qquad \text{as } k \to \infty$$

where C is some strictly positive constant (see Brockwell and Davis (1987) for a proof). As the lag k increases the distance between the autocorrelation function reduces to zero.

Because processes with a unit root are only a special case of nonstationary processes, ARFIMA models provide a more general alternative to standard Dickey-Fuller tests, whose adequacy has been questioned in several instances (see for example Cochrane (1991) and Campbell and Perron (1991)).

Indeed Dickey-Fuller tests have low power when the true underlying process is fractionally integrated (see Diebold and Rudebush (1991)). This critique motivates the use of fractionally integrated models to test for the presence of bubbles.

The order of integration and the other ARFIMA parameters can be estimated either through a two-stage procedure or through maximum likelihood. In the former case, one first estimates d, then filters the series with the fractional version of the standard differencing procedure and finally uses standard techniques to estimate the ARMA parameters from the filtered data (see Geweke and Porter-Hudak (1983)).

Alternatively maximum likelihood methods can be used either in approximate form (McLeod and Hipel (1985), Fox and Taqqu (1986)) or in exact form (Sowell (1992)). Sowell's exact method is used in this study.

To conclude this section, we summarize for later reference two important results on the algebra of fractionally integrated series proved by Granger (1980b). First if a fractional difference operator of order d' is applied to a variable integrated of order d (in symbols $X_t \sim I(d)$), then the resulting process, $Y_t = (1-L)^{d'} X_t$, is integrated of order $I(d+d')$. Second, given two processes, $X_{1,t} \sim I(d_1)$ and $X_{2,t} \sim I(d_2)$, their sum

$$W_t = X_{1,t} + X_{2,t}$$

is integrated of order $\max(d_1,d_2)$. Thus a fractional filter of order d' increases by d' the order of integration and the highest order of integration prevails in a sum of two integrated series. As it will be clear later these two properties are crucial in assessing the correctness of the Hamilton and Whiteman test.

III. Cagan's Equation with Fractionally Differenced Variables

Cagan (1956) made an essential contribution to the analysis of inflationary processes by expressing the money demand as

$$M_t/P_t = c \exp(-\delta \pi^*_{t+1}) \tag{3.1}$$

where M_t is nominal money balances at time t, P_t is the price level at time t and π^*_{t+1} is expected inflation at time t+1, while δ and c are constants, the first reflecting the impact of expected inflation, the second summarizing all other effects. In Cagan's original formulation expectations were assumed to be adaptive, but the more recent literature is based on rational expectations. Let the variables in logarithms be indicated by lower case letters and c be normalized to 1, then expected inflation can be expressed as

$$\pi^* = \frac{E[P_{t+1}|\Omega_t] - P_t}{P_t} = E[p_{t+1}|\Omega_t] - p_t ,$$

where $E[\bullet|\Omega_t]$ denotes mathematical expectation conditional on the information set at time t, Ω_t. The money demand equation (3.1) can then be rewritten as

$$m_t - p_t = -\delta \; (E[p_{t+1} | \Omega_t] - p_t) \tag{3.2}$$

Adding a money demand disturbance n_t, equation (3.2) can be equivalently expressed as

$$E[p_{t+1} | \Omega_t] - \alpha p_t = x_t + n_t \tag{3.3}$$

where $\alpha = (1-\delta)/\delta$ and $x_t = m_t/(1+\delta) = (1-\alpha)m_t$. 1/ Equation (3.3) can also be obtained as a log-linear approximation to an Overlapping Generation (OLG) model with money. This interpretation of (3.2), as explained in Blanchard and Fisher (1989) Chapter V, does not place any restriction on the coefficient α, which, according to Cagan's formulation, was between 0 and 1.

It is crucial in the study of inflation, to stress that monetary policy cannot be isolated from fiscal policy and is ultimately a direct consequence of it. As explained in the next subsection this notion is fundamental in defining the stochastic process for money supply as a long memory process and in addition justifies the presumption that money is exogenous to prices as required by the testing procedure.

1. Monetary policy and fiscal regime

To get testable implications from (3.3) we need to define the stochastic process governing money supply and the money demand disturbance. Furthermore, to implement the test it is required that the driving variable is exogenous. A rather general specification was given by Hamilton and Whiteman (1985)

$$(1 - L)^d \; x_t = A(L)\epsilon_{1,t} + B(L)\epsilon_{2,t}$$

$$(1 - L)^d \; n_t = R(L)\epsilon_{1,t} + S(L)\epsilon_{2,t} \tag{3.4}$$

where the white noise innovations $\epsilon_{i,t}$ $i=1,2$ are jointly fundamental for the bivariate process (x_t, n_t), d is fractional, and A(L), B(L), R(L) and S(L) are polynomial in the lag operator with mean square converging terms. One can think of x_t as a variable observed by the econometrician, in the sense that time series of past realizations are available, and n_t as unobservable by the econometrician (because no data are available), but observable by the agents. For example n_t can be interpreted as variables other than fundamentals that enter agents' forecasts. The first equation in system

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1/ This form of the expectational equation has been studied in a number of papers and in different contexts, for example, asset pricing and exchange rate theory.

(3.4), often called a feedback rule, models a rather broad class of monetary policies (see Sargent (1987), Chapter 17). For example, when $A(L) = 1$, $B(L) = 0$, and $d = 1$ the monetary rule is a random walk. In general it asserts that the monetary authority reacts to unexpected shocks in the economy, described by the disturbances ϵ_i, by choosing $A(L)$ and $B(L)$. This choice depends on the objective function of the monetary authority. 1/

In case of chronic high inflation or hyperinflation monetary policy is unlikely to be directed at stabilizing output growth, rather it is dictated by the need to finance budget deficits. Let's assume that the government budget constraint is represented by

$$G_t + (1 + r_{t-1})B_{t-1} - T_t = (M_t - M_{t-1})/P_t + B_t \qquad (3.5)$$

The identity (3.5) asserts that the difference between total government spending (the sum of real expenditures G_t and interest payments $(1+r_{t-1})B_{t-1}$) and revenues T_t is covered by money printing $M_t - M_{t-1}$ (which extracts a seignorage equal to $(M_t - M_{t-1})/P_t$) or by issuing bonds B_t bearing a real interest rate r_t.

A major implication of (3.5) is that inflation and fiscal deficit are not necessarily temporally related because governments can resort to borrowing. So if a government follows a Ricardian Rule, i.e., is committed to finance its debt exclusively by issuing bonds, $M_t - M_{t-1} = 0$, deficits are not inflationary, as far as the commitment is credible, i.e., if the future stream of expenditures equals the future stream of revenues. Stated differently government debts are not inflationary when they are temporary, so that the budget is balanced in a present value sense. In reality, however, the fiscal authority sets G_t and T_t, and the monetary authority the decides to cover the debt by money creation or by borrowing, or by a combination of the two. The rate of inflation in the short term is a consequence of this choice.

Obviously borrowing cannot be unlimited. When the debt reaches such a level that the future stream of surpluses required to offset it, cannot be supported by the economy, private (domestic and foreign) agents refuse to

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1/ In many models (see e.g. Sargent (1989)) the objective of the authorities is believed to be the minimization of the mean squared error of real GNP growth rate y_t around some predetermined target level y^*, in symbols

$$\min E[(y_t - y^*)^2]$$

where y_t is linked to the real money balances through a simple portfolio balance schedule of the form

$$y_t = m_t - p_t + n_t$$

subscribe to government securities, no matter how high the interest rate is. In practice when the government can only resort to massive confiscation to meet its commitment, agents are afraid that it is an easier (politically more feasible) alternative to "expropriate" bondholders. In anticipation of this occurrence agents require higher and higher real interests to the point that the monetary authority has no choice but to monetize the debt. In the extreme, all the deficit is financed by money printing, that is by an immediate inflation tax. In this event $B_t = 0$. Clearly between Ricardian Rule and full monetization there is a whole range of intermediate.
These issues have been treated in a vast literature (Sargent (1986) contains an excellent review), but rarely have been explicitly mentioned in empirical studies on bubbles in hyperinflation. In particular it should be made clear that Cagan's equation holds for a given fiscal policy which forces the monetary authority to resort prevalently to money creation to finance expenditures.

This lengthy digression explains the choice of the functional form (3.4) for the money supply with a twofold rationale. One, that in periods of chronic high inflation only small increases in B_t are feasible, i.e., $B_t - B_{t-1} \approx 0$, so persistent deficits are almost fully monetized. Two, fiscal imbalance is the product of relatively stable historical conditions and unlikely to change suddenly, hence deficits must be modelled by a highly persistent stochastic process, like (3.4), and so does the monetary rule. Furthermore the substantial dependence of monetary policy from the fiscal regime determines the exogeneity of money supply with respect to the price level which is an implicit prerequisite for the Hamilton and Whiteman test.

2. The solution to the expectational equation and its testable implications

Equation (3.3) with variables specified as in (3.4) can be solved using the z-transform method conjecturing a solution of the form:

$$p_t = \xi(L)\epsilon_{1,t} + \kappa(L)\epsilon_{2,t} \tag{3.6}$$

The appendix contains a simple example of the z-transform method with a fractionally integrated driving variable. The same type of solution can be devised in the present, more general, case, following Hamilton and Whiteman (1985).

$$(1-\alpha L)(1-L)^d p_t = \{\xi_0(1-L)^d + LA(L) + LR(L)\}\epsilon_{1,t} + \{\kappa_0(1-L)^d + LB(L) + LS(L)\}\epsilon_{2,t}. \tag{3.7}$$

Depending on the value of α, the constants ξ_0 and κ_0 can be determined by the requirement that the functions $(1-z)^d\xi(z)$ and $(1-z)^d\kappa(z)$ must be analytic on the unit circle. For $|\alpha| > 1$ this leads to:

$$\xi_0 = -(1-1/\alpha)^{-d}\{\alpha^{-1} A(1/\alpha)+1/\alpha\ R(1/\alpha)\}$$

$$\kappa_0 = -(1-1/\alpha)^{-d}\{\alpha^{-1} B(1/\alpha)+1/\alpha\ S(1/\alpha)\}. \tag{3.8}$$

For $|\alpha| \leq 1$, these conditions are not required for analyticity, but hold if the parsimony principle is invoked, that is if model specification involves the minimum number of explanatory variables.

The solution (3.6)-(3.8) to equation (3.3), valid for all processes x_t and n_t that can be represented in terms of square summable operators A(L), B(L), R(L), S(L) is called the fundamental solution for it depends on the driving variable only.

By contrast, solutions to (3.3) depending on other--possibly completely independent variables--are referred to as bubble or sunspot solutions. Following Hamilton and Whiteman (1985) and Burmeister, Flood and Garber (1983), the particular solution as a function of any finite number of white noises $\eta_{i,t}$ $i=1,2,\ldots,m$, completely unrelated to (x_t,n_t), and of the fundamental solution p^*_t is

$$y(1-\alpha L)(1-L)^d p_t = (1-\alpha L)(1-L)^d p^*_t+Q_1(1-L)^d\eta_{1,t}+\ldots+Q_m(1-L)^d\eta_{m,t}, \tag{3.9}$$

for any values Q_i. The condition $Q_i \neq 0$ for some i, implies the existence of a bubble or sunspot equilibrium.

The essence of the Hamilton and Whiteman test is to estimate the difference in the order of integration of the two series. From (3.7) we divide both sides by $(1-\alpha L)$:

$$(1-L)^d p_t = (1-L)^d p^*_t+(1-\alpha L)^{-1}[Q_1(1-L)^d\eta_{1,t}+\ldots+Q_m(1-L)\eta_{m,t}]+K\alpha^t, \tag{3.10}$$

where K is an arbitrary constant. If $K = Q_i = 0\ \forall i$, i.e., in the absence of bubbles or sunspots, both sides of (3.9) are integrated of the same order. Otherwise, the nonstationary terms $(1-\alpha L)^{-1}\eta_{i,t}$, by the algebra of integrated series, would render the right hand side integrated of an order higher than d.

As discussed by Hamilton and Whiteman (1985) this difference in the order of integration is the only testable hypothesis implied by nonfundamental solutions because the parameters of the model (3.3)-(3.4) are observationally equivalent in the presence or absence of bubbles or other extraneous influences. In addition this test does not rely on the assumed stochastic process for the nonstationary component. In this respect the test is general, while results obtained in previous studies depend on the arbitrary choice of the bubble or sunspot process.

IV. Detecting Nonfundamental Influences

This section is devoted to some basic issues arising in testing for bubbles or sunspots in rational expectations models. The starting point is Evans' critique of the Hamilton and Whiteman test which epitomizes the confusion arising in the analysis of nonstationary processes when standard techniques are employed.

Evans (1991) defines a bubble process of the form:

$$B_{t+1} = (1+r)B_t u_{t+1} \qquad B_t \leq \alpha \tag{4.1}$$

$$B_{t+1} = \left[\delta + \frac{1}{\pi}(1+r)\theta_{t+1}\left(B_t - \frac{\delta}{1+r}\right)\right]u_{t+1} \qquad B_t > \alpha,$$

where δ and α are positive parameters which satisfy $0 < \delta < (1+r)\alpha$, u_{t+1} is an identically and independently distributed exogenous positive random variable with unit expectation, and θ_{t+1} is an identically and independently distributed exogenous Bernoulli process, unrelated to u, so that:

$$\theta_{t+1} = \begin{cases} 1 & \textit{with prob. } \pi \\ 0 & \textit{with prob. } (1-\pi) \end{cases} .$$

Note that B_t is always positive, grows at rate 1+r until it exceeds the value α, where it starts to explode at a mean rate $(1+r)/\pi$, subject to a probability $1-\pi$ of bursting and reverting to a mean value of δ. The average duration, the peak, the frequency of bursts, etc., depend on the values of the parameters. Evans (1991) asserts that this "nonstationary" bubble cannot be detected by unit root tests. In a Montecarlo study, in fact, it appears that the Dickey-Fuller test regularly rejects the hypothesis of nonstationarity for 100 realizations of B_t. Evans (1991) thus concluded that the Hamilton and Whiteman's idea although theoretically valid, has limited practical application.

What this critique highlights is the inadequacy of the statistical methods based on integer orders of integration. 1/ The crucial feature of Evans' process is its periodic collapse. When added to the fundamental solution, it affects only the high-frequency components of the spectral density. The nonstationarity of a series is directly linked to its spectral density, and in particular to its slope near the origin; hence a "bubble" is not detectable unless it alters in a significant way the lower frequency terms. A look at Chart 1 gives a visual insight to this argument. Realizations of B_t with 400 observations for different values of the parameters are depicted. It is evident that this "bubble" bursts fast, when compared to the length of the series, thus its effect is notable only in the short-cycle components of the spectrum. The Dickey Fuller test, based on integer orders of integration, concentrates on the low frequency components of the spectrum and for this reason it is inadequate to analyze the process suggested by Evans (1991) and all the homogeneous terms that alter the high frequency (short-term) components in the fundamental solution to (3.3). On the contrary, fractionally integrated models provide a viable way of detecting the presence of bubbles or extraneous influences. A formal argument can be introduced by looking at the spectrum of ARFIMA(p,d,q) processes:

$$f(\lambda) = |1 - e^{-i\lambda}|^{-2d} f_u(\lambda),$$

where

$$f_u(\lambda) = (\sigma^2 |\theta \exp(-i\lambda)|^2)/(2\pi \; |\phi \exp(-i\lambda)|^2)$$

is the spectral density of the ARMA(p,q) process $u_t = (1-L)^d X_t$; taking the natural logarithm, adding and subtracting $\log(f_u(0))$, we obtain:

1/ Hall and Sola (1993) and Blackburn and Sola (1993), have resorted to the Markov regime switching proposed by Hamilton (1988, 1990), to overcome Evans' critique. While this approach is based on premises quite different from the ARFIMA models, it nevertheless is well suited for detecting the presence of Evans' processes.

$$\ln f(\lambda) = \ln f_u(0) - d \ln|1-e^{-i\lambda}|^2 + \ln[f_u(\lambda)/f_u(0)].$$ 1/

Therefore, the order of differentiation d is essentially the negative slope of the log spectrum close to the origin, which is equivalent to asserting that the order of integration has a more pronounced dependence on the low frequency components.

To verify this theoretical proposition, we estimated the order of integration of a number of synthetic processes B_t with the same parameters Evans uses in his paper. In all cases the estimated order of integration was around 0.3, which implies first, that this type of process is stationary, and second, that it can be detected by estimating the difference in the order of integration as Hamilton and Whiteman suggest. This conclusion is obviously not limited to the specific form of process Evans

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1/ This expression is also the basis for the estimation of the parameter d. Following Geweke and Porter-Hudak (1983), substituting to the frequencies λ, $0 < \lambda < \pi$, the Fourier frequencies $\omega_j = 2\pi j/n \in (0,\pi)$ and adding the log of the periodogram $I_n(\omega_j)$ to both sides, we find:

$$\ln I_n(\omega_j) = \ln f_u(0) - d\ln|1-e^{-i\omega j}|^2 + \ln[I_n(\omega_j)/f(\omega_j)] + \ln[f_u(\omega_j)/f_u(0)],$$

which has the form of an OLS regression:

$$Y_j = a + \beta x_j + \eta_j.$$

$Y_j = \ln I_n(\omega_j)$, $a = \ln f_u(0)$, $x_j = \ln|1-e^{-i\omega j}|^2$, $\ln[I_n(\omega_j)/f(\omega_j)]$ represents the normally distributed error term η and $\ln[f_u(\omega_j)/f_u(0)]$ becomes negligible as the attention concentrates on harmonic frequencies near zero. The estimator of d, therefore, will be given by:

$$\bar{d} = \frac{-\sum_{i=1}^{m} (x_i - E(x))(Y_i - E(Y))}{\sum_{i=1}^{n} (x_i - E(x))^2},$$

which, according to Geweke and Porter-Hudak simulations, for $m = n^{1/2}$ is the asymptotically normal best linear unbiased estimator.

considers, but to all processes that are stationary and nonstationary with a nonzero order of integration. 1/

As a final remark, it is certainly conceivable that in speculative markets, a process like B_t could be started by news hitting the market, but it would be much less likely that a hyperinflation process, which is a long and sustained phenomenon, could be attributed to this sort of rapidly bursting homogeneous component. Data on hyperinflation do not show sequences of rapidly collapsing price explosions. Inflation, on the contrary, is persistent rather than intermittent. Our aim in the next section is to verify whether it is more persistent than money supply increases.

V. The Empirical Analysis

In essence, the empirical analysis consists of estimating the fractional order of integration of the money supply and the consumer price level. If no bubbles or sunspots are present the order should be approximately equal.

The estimation was carried out using a Fortran program written by Fallaw Sowell at Carnegie Mellon and based on Sowell (1993). 2/ The data were obtained from International Financial Statistics, published by the International Monetary Fund. This source provides a homogeneous definition for both variables across countries and therefore allows a meaningful comparison of the results. It is important to stress that the money series refer to narrow money (line 34 in the IFS) and the price level is the consumer price level (line 64 in the IFS).

The data span different periods, but are always on a monthly basis. For Argentina and Peru, they start in January 1971 and end in December 1989, while for Bolivia, Brazil and Chile they extend only through December 1987,

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1/ A word of caution is in order, namely, that the empirical method used to estimate the order of integration of B_t is important; in fact, a two-stage procedure, such as proposed by Geweke and Porter-Hudak (1983), is not well suited for this purpose. The reason is that it attaches too much importance to lower frequency components of the spectrum, while B_t has more influence on short cycles. In fact, to verify empirically this claim in a Monte Carlo study, we estimated with the Geweke and Porter-Hudak procedure the fractional difference parameter of two hundred realizations of the bubble processes proposed by Evans (1991). The results showed that d was never significantly different from zero, whereas its true value is 0.3.

2/ We introduced a modification in the original version by resorting to a different subroutine for calculating the roots of autoregressive polynomials. It was developed at the Department of Astrophysics of the University of Chicago and employs Muller's method. The calculations were executed on a Sun Spark 2 Unix workstation.

February 1986, and June 1985 respectively. The data on the Former Socialist Federal Republic of Yugoslavia, cover the period January 1975-June 1990.

The results are shown in Tables 1-6. For each country are reported the results relative to all models for which the maximum likelihood optimization converged to meaningful values and the sensitivity to initial conditions was not extreme. The identification procedure for ARFIMA models is still a debated question so the tables offer a detailed account of all the results, not only those relative to the model identified by any controversial criterion.

Specifically, for each model the tables report the estimated value of d with t-statistics in parenthesis; the estimated autoregressive and moving average parameters (when present) in the columns AR and MA, starting from the first lagged component; the ratio between the variance of the predicted values and the variance of the data in the column σ/var (which is a sort of R^2 statistic); the Akaike Information Criterion (AIC); the Schwartz Information Criterion (SIC) and finally the log-likelihood at optimum (MAX. LIK).

Moreover in the tables negligible sensitivity designates a situation where changing the initial values affects only slightly (plus o minus 10 percent) the t-test values; little sensitivity means that the parameters estimates exhibits changes in the second decimal digit, and sensitivity to initial condition refers to the case where the maximization algorithm finds local maxima so that parameter estimates change in value and significance. For the latter the tables report the model with the highest log-likelihood.

Both the AIC and the SIC strongly point at the ARFIMA(1,d,0) as the preferred model. In no case and for none of the models reported in the tables, the estimates indicate that the variables have different levels of nonstationarity. Furthermore the 95 percent confidence intervals indicate that difference in the orders of integration of money supply and prices are never very significant. This evidence suggests that sunspot or bubble equilibria can be excluded based on the Whiteman and Hamilton procedure. However, it is true that the orders of integration of the price level and money are not always exactly the same, which means that the short-cycle components of money supply and price levels are different.

In what follows a detailed account of the results for each country will be presented.

Former Socialist Federal Republic of Yugoslavia. This is one of the two countries where the difference in the order of integration between the money supply and price levels is negligible. The AIC and SIC values leave few doubts about the identification of the ARFIMA(1,d,0) model. Further, the other models yield estimates of d which are similar to the ARFIMA(1,d,0). The only exception is the ARFIMA(1,d,2) for price level.

Therefore, the evidence against the existence of bubbles is strong.

Peru. The results for Peru are analogous, except for the somewhat greater sensitivity to initial conditions registered for the model ARFIMA(1,d,0) relative to the price level. Otherwise, the estimates of d for both series are extremely close in all models.

Chile. The model ARFIMA(1,d,0) identified by the AIC and the SIC for both series exhibits an estimated d equal to 0.31 for the money supply and to 0.4 for the price level. The result does not change substantially when we exclude from the initial sample period the years 1971, 1972, and 1973, i.e., the period before the military took power.

This difference in the orders of integration is not substantial: the hypothesis that they are different would not be accepted at conventional significance levels. However, we want to draw attention to the fact that, for Chile, the estimates of d vary across models. Further, the AIC, and especially the SIC, for the model ARFIMA(2,d,0) are close to those for ARFIMA(1,d,0), so the identification procedure in this case leaves some degree of uncertainty. The estimates of d in the ARFIMA(2,d,0) models are practically identical: 2.48 for price level and 2.47 for money supply.

In conclusion, even for Chile the evidence in favor of the "no bubbles" hypothesis is strong, although not as categorical as in the first two cases.

Bolivia. A strict adherence to the AIC and SIC again points to the ARFIMA(1,d,0) representation. The estimates of d again differ significantly, but not enough to establish the presence of bubbles. Moreover, although the ARFIMA(2,d,0) does not fully capture the behavior of prices, it is adequate to describe the dynamics of money supply. In this case the estimate of d is 2.46 much closer to the value 2.43 obtained with the ARFIMA(1,d,0), the most reliable model for the price level.

Argentina. Here the difference in the estimated d values is more pronounced. The ARFIMA(1,d,0) representation, selected by the AIC and SIC, shows 2.47 for the price level and 2.35 for the money supply. Moreover, the identification methodology does not suggest other viable models. Therefore, it follows that a difference in the order of integration is likely.

Brazil. This is the country where the difference in the estimated order of integration for the ARFIMA(1,d,0) model is most significant. On this basis, the existence of a rapidly collapsing bubble process cannot be excluded, although there is no evidence of difference in the order of stationarity.

However, unlike for Argentina, the strength of this conclusion is mitigated by the fact that for money supply, the value of AIC and SIC relative to the ARFIMA(2,d,0) (equal to 473 and 464 respectively), are comparable to the values relative to the ARFIMA(1,d,0), (439 and 433 respectively). The estimated d parameter in the ARFIMA(2,d,0) representation is 2.43, close to the value of 2.48 obtained for the price level.

What is the general picture emerging from the econometric analysis? We can strongly reject the existence of bubbles that alter the degree of stationarity of the fundamental solution. But the mixed results we obtain for Argentina and Brazil need further interpretation. Two lines of reasoning will be presented. One focuses on the credibility of fiscal reforms, the other on the legal mechanisms devised to shield certain groups from the effects of inflation.

1. <u>Fiat money and the present value of future budget deficits</u>

Sometimes the literature on hyperinflation concentrates on econometrics and pays little attention to an aspect of extreme relevance highlighted in Chapter 3 of Sargent (1986). The same amount of high-powered money in circulation in different fiscal regimes, leads to different inflation rates. For example, in Austria the hyperinflation process after World War I was stopped by the change in fiscal regime following the accord on the reform of the financial system and the fiscal policy, signed on October 2, 1922, between the Austrian Government and the Council of the League of Nations. The sudden stop in hyperinflation was achieved despite a six-fold increase in high-powered money between August 1922 and December 1924. 1/ No currency reform was undertaken.

Therefore, the nature of Cagan's equation should be interpreted as expressing a relation between money supply and price levels for a given fiscal rule. Substantial changes in fiscal regimes alter the relationship between money and inflation and this must be reflected in the econometric analysis.

The two countries where the absence of bubbles is unequivocal, Peru and Former Socialist Federal Republic of Yugoslavia, are those where fiscal reform was never seriously pursued in the period examined.

Chile achieved a substantial stabilization pursuing a steady fiscal discipline without substantial variations, which nevertheless took many years to be completed.

The other three countries have a history of economic and political turmoil with frequent shifts from civilian to military rules and vice-versa, which gave rise to substantial uncertainty over fiscal policies and the credibility of announced stabilization programs. For Argentina and Brazil the consequence has been a chronic high rate of inflation.

The key feature of fiscal reform consists in credibly redefining the general strategy of tax collection, transfer and expenditures. Isolated actions or unenforceable declarations of intent (the promise to fight tax

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1/ See Sargent (1986), Chapter 3, for a more detailed account.

evasion is a classic example) should not be regarded as reforms. Actually they tend to have the opposite effect, as the public realizes that they are a symptom of impotence, rather than determination to tackle the roots of the problems.

Therefore, the difference in the order of integration detected through the econometric analysis is likely to reflect the correct perception that an inadequate fiscal stance increases the present value of future deficits and substantially lowers the value of unbacked (outside) money in circulation.

2. Nonstate contingent contracts and the momentum of inflation

A protracted period of high inflation almost inevitably brings a demand for protection. Individual agents or, more likely, coagulated interest groups try to shield themselves from the inflation tax. The best known and widely used device is wage indexation. Indexed government bonds, are another example. Pricing strategies by firms with some market power, produce important effects.

The most important theoretical contribution on persistent inflation were originated by Taylor (1979) and Phelps and Taylor (1977) and focus on staggered wage contracts. Inflation maintains a momentum because agents are forced to bargain multiperiod contracts, which necessarily reflect the expectations over relevant variables at the time they are negotiated. In addition the overlapping of collective contracts in different sectors of the economy contributes to the sluggishness in inflation.

In general all nonstate contingent mechanisms of protection are a source of persistence in inflation. Since these phenomena are independent of monetary policy, the econometrician should find a slight difference in the orders of integration.

VI. Summary and Conclusions

The objective of this paper is to test the hypothesis that high inflations are caused by self-fulfilling expectations, which in the economic literature have been identified with the existence of "bubbles," or by extraneous influences called "sunspots". Mathematically, these effects are depicted by the homogeneous term in the solution to the expectational equation (3.3).

As stressed by Hamilton and Whiteman (1985) the presence of a homogeneous term can only be verified by analyzing the difference in the dynamics of price level and money supply as reflected in the orders of integration. This paper was aimed at testing the significance of this difference for a number of countries during the 1970s and part of the 1980s, based on recent econometric advances in the theory of fractionally integrated processes. This methodology allows one to detect differences in spectral components of price level and money supply overcoming Evans' critique.

Our results show that the difference in order of integration is in general small, although in a few cases it is not negligible: for most countries examined, the AIC and SIC point at an ARFIMA(1,d,0) model for both price levels and money supplies, although the ARFIMA(2,d,0) process is also acceptable in some cases. Most estimates of d lie in the interval [2.45, 2.49] indicating that the second differences of the series are stationary.

These results are similar in spirit--though not in methodology--to related results in the recent literature on hyperinflation including Phylaktis and Taylor (1993), who use the same data set, and employ standard co-integration analysis. Blackburn and Sola (1993) on the other side find that for Argentina, the presence of a rapidly bursting bubble cannot be rejected.

So it seems that, overall, the empirical evidence against persistent self-fulfilling expectations or sunspots is strong. What cannot be excluded in some cases is the sporadic temporary divergence between the money supply and price level short-cycle dynamics.

This phenomenon might be attributed on one side to stabilization programs based on isolated actions, and not on a substantial shift in fiscal regime, on the other to nonstate contingent protection from inflation accorded to some special groups.

An example of the Z-Transform Method for the Solution of Expectational Equations with Fractionally Integrated Variables

In equation (3.1) we assume that the money supply follows a fractional noise

$$m_t = (1-L)^{-d}\epsilon_t = \sum_{j=0}^{\infty} \frac{\Gamma(j+d)}{\Gamma(j+1)\Gamma(d)}\epsilon_{t-j} = F(L)\epsilon_t \tag{A.3}$$

The crucial condition for deriving a solution is the convergence in mean square of the terms f_i, i.e., (see Appendix):

$$\sum_{i=0}^{\infty} f_i^2 < \infty$$

which is satisfied for $-1 < d < .5$.

According to the z-transform method described in Whiteman (1983) and extended in a more general setting in Hamilton and Whiteman (1985), the solution for p must belong to the same space as the driving process m, in this case the Hilbert space generated by linear combinations of ϵ_t. Actually Whiteman (1983) stated the Solution Principle in four tenets:

1. The driving process must be a covariance stationary stochastic process with an explicit Wold representation.

2. Expectations are formed rationally in the sense of Muth (1961) and the predictors can be found through the Wiener-Kolmogorov formulas.

3. Solutions lie in the space generated by square summable time independent linear combinations of the driving process.

4. The rational expectation hypothesis holds for all realizations of the driving process.

Hamilton and Whiteman (1985) show that the z-transform technique is applicable to any dynamic behavior representable in terms of square summable operators.

The general form of the solution will have the non-normalized Wold representation:

$$p_t = \sum_{i=0}^{\infty} (c_i L^i)\epsilon_t = C(L)\epsilon_t \, , \tag{A.4}$$

where the c_i's are mean square convergent. The z-transform method allows to find a form for C(z) in terms of F(z) and the parameter of the model, provided that F(z) and C(z) are both holomorphic on the open unit disk $|z| < 1$. Since we implicitly assume that:

$$\Omega_t = \{\epsilon_t, \epsilon_{t-1}, \epsilon_{t-2}, \ldots\} \quad \Rightarrow \quad \Omega_t \supset \Omega_{t-1} \supset \Omega_{t-2} \supset \Omega_{t-3} \ldots\ldots,$$

the Wiener-Kolmogorov predictor is:

$$E(p_{t+1} \mid \Omega_t) = c_1\epsilon_t + c_2\epsilon_{t-1} + \ldots..$$

Putting in evidence the reciprocal of the lag operator, the above expression becomes:

$$E(p_{t+1} \mid \Omega_t) = L^{-1}(c_1\epsilon_{t-1} + c_2\epsilon_{t-2} + \ldots..) = L^{-1}[C(L)-c_0]\epsilon_t.$$

Substituting this result in 5.2.1, we get:

$$[L^{-1}[C(L)-c_0]\epsilon_t - \alpha C(L)\epsilon_t = F(L)\epsilon_t ,$$

which implies:

$$(1-az)C(z) = zF(z)+c_0 ,$$

and thus:

$$C(z) = (zF(z)+c_0)\ (z-\alpha)^{-1}.$$

For a solution to exist, C(z) has to be holomorphic in $|z| < 1$, which is true if, and only if, $|\alpha| \le 1$. This is the case we treat later. When $|\alpha| > 1$, for C(z) to be analytic on the open unit disk, we have to show that the singularity at $z=\alpha^{-1}$ is removable, that is, we have to show that:

$$\lim_{z \to \alpha^{-1}}(1-\alpha z)C(z) = 0 \ \rightarrow\ \alpha^{-1}F(\alpha^{-1}) < \infty. \tag{A.5}$$

Applying Stirling's formula as $i \rightarrow \infty$ $f_i \approx i^{d-1}/\Gamma(d)$ and hence for $d \in (0,1)$:

$$F(\alpha^{-1}) = 1+\frac{1}{\Gamma(d)}\sum_{i=1}^{\infty}\frac{1}{i^{1-d}\alpha^{i}} < 1+\frac{1}{\Gamma(d)}\sum_{i=1}^{\infty}\frac{1}{\alpha^{i}} = 1+\frac{1}{\Gamma(d)\,(1-\alpha)} \,. \tag{A.6}$$

As a consequence, $c_0 = \alpha^{-1}F(\alpha^{-1})$ or $c_0 = \alpha^{-1}(1-\alpha^{-1})^{-d}$. The unique solution for p_t in the latter case, according to the Solution Principle, is given by:

$$p_t=\frac{LF(L)-\alpha^{-1}F(alHxa^{-1})}{1-\alpha L}\epsilon_t=(1-\alpha L)^{-1}\left[\sum_{i=0}^{\infty}\frac{\Gamma(i+d)}{\Gamma(i+1)\Gamma(d)}L^{i+1}\epsilon_t-\alpha^{-1}F(\alpha^{-1})\,\epsilon_t\right], \tag{A.7}$$

which can be re-expressed as:

$$\begin{aligned}(1-\alpha L)\,p_t &= L\sum_{i=0}^{\infty} f_i\epsilon_{t-i}-\frac{1}{\alpha}\sum_{i=0}^{\infty} f_i\alpha^{-i}\epsilon_t \\ &= (L-\alpha^{-1})\sum_{i=0}^{\infty} f_i\,(\epsilon_{t-i}+\alpha^{-i}\epsilon_t)\end{aligned} \tag{A.8}$$

$$p_t = \frac{\alpha L-1}{\alpha\,(1-\alpha L)}\sum_{i=0}^{\infty} f_i\,(\epsilon_{t-i}+\alpha^{-i}\epsilon_t)\,. \tag{A.9}$$

This last result shows that the price is the sum of two components, i.e.,

$$p_t = -\nabla^{-d}\,\alpha^{-1}\epsilon_t - \eta_t, \tag{A.10}$$

where $\eta_t = c_0\epsilon_t$, and from the algebra of the integrated series we conclude that $p_t \sim I(d)$, that is, the same order of integration as m_t. Analogously, when $|\alpha| < 1$, i.e., when the expectation of future price changes has an impact greater than the one on current prices, the restriction that $C(z)$ and $F(z)$ must be analytic does not specify a value for c_0 and no unique solution can be found. The general solution has the form

$$p_t = \frac{LF(L)+c_0}{1-\alpha L}\,\epsilon_t = (1-\alpha L)^{-1}\left[c_0\epsilon_t+\sum_{i=0}^{\infty}\frac{\Gamma(i+d)}{\Gamma(i+1)\Gamma(d)}\epsilon_{t-1-i}\right]. \tag{A.11}$$

The difference from (A.9) is the more specific form of the solution, which is an ARFIMA(1,d,0) with AR parameter α. (A.10) and (A.11) are usually referred to as the fundamental, or bubble-free, solution.

Table 1. Argentina

Price Level

ARFIMA (1, d, 0) (No sensitivity to initial conditions)

d	AR	MA	σ/var	AIC	SIC	MAX.LIK
2.47	-0.64		0.24	396	390	200.4
(12.04)	(-12.03)		(10.08)			
conf. intvl						
±0.08						

ARFIMA (1, d, 1) (Little sensitivity to initial conditions)

d	AR	MA	σ/var	AIC	SIC	MAX.LIK
2.24	-0.55	0.99	0.13	529	519	268.00
(2.18)	(5.18)	(67.9)	(10.60)			
conf. intvl						
±0.21						

ARFIMA (2, d, 0) (Negligible sensitivity to initial conditions)

d	AR	MA	σ/var	AIC	SIC	MAX.LIK
2.46	-1.04		0.16	485.5	475.2	245.7
(8.28)	(-16.93)		(10.15)			
	0.59					
conf. intvl	(11.08)					
±0.11						

ARFIMA (2, d, 1) (Negligible sensitivity to initial conditions)

d	AR	MA	σ/var	AIC	SIC	MAX.LIK
2.21	-0.67	0.98	0.13	533	519	270.8
(1.74)	(-4.72)	(63.60)	(10.91)			
	0.16					
conf. intvl	(2.44)					
±0.23						

ARFIMA (2, d, 2) (Little sensitivity to initial conditions)

d	AR	MA	σ/var	AIC	SIC	MAX.LIK
2.39	-1.04	0.48	0.12	535	518	272.7
(3.72)	(-8.82)	(3.51)	(10.24)			
	0.42	-0.55				
conf. intvl	(7.20)	(-3.95)				
±0.20						

Table 1. Argentina

Money Supply

ARFIMA (1, d, 0) (No sensitivity to initial conditions)

d	AR	MA	σ/var	AIC	SIC	MAX.LIK
2.35 (4.36)	-0.15 (6.07)		0.36 (10.71)	338	331	171.31
conf. intvl ±0.16						

ARFIMA (1, d, 1) (No sensitivity to initial conditions)

d	AR	MA	σ/var	AIC	SIC	MAX.LIK
2.35 (4.66)	-0.05 (0.52)	1.01 (48.04	0.23 (10.23	429	419	217.9
conf. intvl ±0.15						

ARFIMA (2, d, 0)

d	AR	MA	σ/var	AIC	SIC	MAX.LIK
2.46 (9.15)	-0.59 (-7.90) 0.35 (5.71)		0.32 (10.13)	365	355	185.8
conf. intvl ±0.10						

ARFIMA (0, d, 2) (No sensitivity to initial conditions)

d	AR	MA	σ/var	AIC	SIC	MAX.LIK
2.35 (4.59)		1.05 (10.10) 0.05 (0.55)	0.24 (10.59)	429	419	217.9
conf. intvl ±0.15						

ARFIMA (2, d, 2) (Sensitivity to initial conditions)

d	AR	MA	σ/var	AIC	SIC	MAX.LIK
2.37 (5.12)	0.94 (10.4) 0.31 (3.26)	2.00 (125.92) 1.00 (62.9)	0.23 (10.24)	429	413	217
conf. intvl ±0.14						

Table 1. Argentina

ARFIMA (2, d, 2) (Sensitivity to initial conditions)

d	AR	MA	σ/var	AIC	SIC	MAX.LIK
2.37	-0.03	1.00	0.23	428	414	218
(4.01)	(-0.28)	(51.82)	(10.06)			
	0.03					
conf. intvl	(0.39)					
±0.18						

Table 2. Brazil

Price Level

ARFIMA (0, d, 0) (No sensitivity to initial conditions)

d	AR	MA	σ/var	AIC	SIC	MAX.LIK
2.48 (17.5)	-0.45 (-6.22)		0.11 (9.6)	861	855	433
conf. intvl ±0.05						

ARFIMA (1, d, 1) (No sensitivity to initial conditions)

d	AR	MA	σ/var	AIC	SIC	MAX.LIK
2.49 (26.78)	0.15 (0.98)	-1.22 (6.7)	0.06 (3.28)	887	867	467
conf. intvl ±0.03						

ARFIMA (2, d, 0) (Negligible sensitivity to initial conditions)

d	AR	MA	σ/var	AIC	SIC	MAX.LIK
2.49 24.76	-0.58 (-8.10) 0.33 (4.55)		0.10 (9.53)	879	869	442
conf. intvl ±0.40						

ARFIMA (2, d, 1) (negligible sensitivity to initial conditions)

d	AR	MA	σ/var	AIC	SIC	MAX.LIK
2.24 (18.22)	0.28 (3.12)	0.98 (38.24) -0.18 (-2.56)	0.09 (8.43)	889	876	448.9
conf. intvl ±0.20						

ARFIMA (1, d, 2) (Sensitivity to initial conditions)

d	AR	MA	σ/var	AIC	SIC	MAX.LIK
2.48 (25.48)	0.82 (11.02)	1.83 (6.7) 1.8 (4.21)	0.03 (2.17)	893	880	450
conf. intvl ±0.04						

Table 2. Brazil

Money Supply

ARFIMA (1, d, 0) (Little sensitivity to initial conditions)

d	AR	MA	σ/var	AIC	SIC	MAX.LIK
2.23 (3.53)	-0.3 (-2.82)		0.66 (9.45)	439	433	221
conf. intvl ±0.13						

AFRIMA (0, d, 1) (No sensitivity to initial conditions)

d	AR	MA	σ/var	AIC	SIC	MAX.LIK
2.14 (2.90)		1.02 (61.47)	0.41 (0.17)	511.2	504.7	257
conf. intvl ±0.10						

ARFIMA (1, d, 1) (Little sensitivity to initial conditions)

d	AR	MA	σ/var	AIC	SIC	MAX.LIK
2.20 (4.65)	0.15 (1.56)	0.98 (59.38)	0.43 (9.41)	511	502	258
conf. intvl ±0.08						

ARFIMA (2, d, 0) (Neglible sensitivity to initial conditions)

d	AR	MA	σ/var	AIC	SIC	MAX.LIK
2.43 (7.72)	-0.20 (-2.51) 0.47 (6.62)		0.53 (9.79)	473.1	464	239.8
conf. intvl ±0.11						

AFRIMA (2, d, 1)

d	AR	MA	σ/var	AIC	SIC	MAX.LIK
2.36 (5.50)	0.33 (3.75) 0.33 (4.00)	0.97 (55.89)	0.40 (9.53)	522.1	510.1	265.4
conf. intvl ±0.13						

Table 2. Brazil

AFRIMA (0, d, 2) (Little sensitivity to initial conditions)

d	AR	MA	σ/var	AIC	SIC	MAX.LIK
2.47		0.27	0.40	520.4	510.8	263.2
(10.70)		(4.75)	(0.94)			
		-0.60				
conf. intvl		(-7.80)				
±0.09						

ARFIMA (1, d, 2) (No sensitivity to initial conditions)

d	AR	MA	σ/var	AIC	SIC	MAX.LIK
2.46	-0.17	0.29	0.40	520	508	264.4
(9.08)	(-1.5)	(3.76)	(9.55)			
		-0.67				
conf. intvl		(-9.25)				
±0.10						

Table 3. Bolivia

Price Level

ARFIMA (1, d, 0) (No sensitivity to initial conditions)

d	AR	MA	σ/var	AIC	SIC	MAX.LIK
2.43 (6.72)	-0.50 (-6.88)		0.24 (10.10)	275	268	139.62
conf. intvl ±0.12						

ARFIMA (0, d, 1) (No sensitivity to initial conditions)

d	AR	MA	σ/var	AIC	SIC	MAX.LIK
2.46 (12.33) conf. intvl ±0.07		1.00 (12.08)	0.14 (8.68)	377	370	190

ARFIMA (1, d, 1) (No sensitivity to initial conditions)

d	AR	MA	σ/var	AIC	SIC	MAX.LIK
2.39 (5.20)	-0.18 (-1.52)	0.99 (49.96)	0.14 (9.70)	378	368	192
conf. intvl ±0.15						

ARFIMA (2, d, 0) (Sensitivity to initial conditions)

d	AR	MA	σ/var	AIC	SIC	MAX.LIK
2.49 (4.65)	-0.70 (-11.62) 0.51		0.17 (9.91)	335	325	170.8
conf. intvl ±0.20	(8.13)					

AFRIMA (2, d, 1) (Sensitivity to initial conditions)

d	AR	MA	σ/var	AIC	SIC	MAX.LIK
2.45 (10.26)	-0.15 (-2.12) 0.15	1.01 (7.48)	0.13 (10.59)	380	367	194.4
conf. intvl ±0.08	(2.03)					

Table 3. Bolivia

ARFIMA (0, d, 2) (No sensitivity to initial conditions)

d	AR	MA	σ/var	AIC	SIC	MAX.LIK
2.39		1.22	0.14	380	370	183
(7.55)		(2.02)	(10.08)			
		0.21				
conf. intvl		(3.55)				
±0.10						

ARFIMA (1, d, 2) (Negligible sensitivity to initial conditions)

d	AR	MA	σ/var	AIC	SIC	MAX.LIK
2.39	0.05	1.26	0.14	378	364	193
(5.9)	(0.21)	(16.25)	(0.36)			
conf. intvl		0.25				
±0.13		(2.9)				

AFRIMA (2, d, 2) (Sensitivity to initial conditions)

d	AR	MA	σ/var	AIC	SIC	MAX.LIK
2.47	-0.34	0.79	0.13	379	363	194.9
(11.90)	(-1.60)	(3.56)	(9.68)			
	0.20	-0.22				
conf. intvl	(3.09)	(-1.05)				
±0.08						

Money Supply

ARFIMA (1, d, 0) (Negligible sensitivity to initial conditions)

d	AR	MA	σ/var	AIC	SIC	MAX.LIK
2.32	-0.32		0.50	133	127	68.9
(3.74)	(-3.02)		(10.06)			
conf. intvl						
±0.17						

ARFIMA (0, d, 1) (No sensitivity to initial conditions)

d	AR	MA	σ/var	AIC	SIC	MAX.LIK
2.19		1.00	0.37	185	174	94.8
(4.41)		(72.19)	(9.73)			
conf. intvl						
±0.08						

Table 3. Bolivia

ARFIMA (1, d, 1) (No sensitivity to initial conditions)

d	AR	MA	σ/var	AIC	SIC	MAX.LIK
2.34 (5.48)	0.30 (3.55)	1.00 (78.41)	0.35 (9.90)	194	184	100.2
conf. intvl ±0.12						

AFRIMA (2, d, 0) (Negligible sensitivity to initial conditions)

d	AR	MA	σ/var	AIC	SIC	MAX.LIK
2.46 (9.21)	-0.27 (-3.4) 0.31 (4.47)		0.45 (10.01)	149	139	77.6
conf. intvl ±0.10						

ARFIMA (0, d, 2) (No sensitivity to initial conditions)

d	AR	MA	σ/var	AIC	SIC	MAX.LIK
2.44 (7.35)		0.58 (11.40) -0.42 (-8.28)	0.35 (9.80)	196	186	101.0
conf. intvl ±0.11						

ARFIMA (1, d, 2) (No sensitivity to initial conditions)

d	AR	MA	σ/var	AIC	SIC	MAX.LIK
2.43 (6.02)	0.05 (0.3)	0.62 (3.36) -0.38 (-2.05)	0.35 (9.77)	194	180	101.1
conf. intvl ±0.14						

Table 4. Chile

Price Level

ARFIMA (1, d, 0) (No sensitivity to initial conditions)

d	AR	MA	σ/var	AIC	SIC	MAX.LIK
2.40 (5.3)	-0.38 (-3.85)		0.3 (9.31)	445	438	224
conf. intvl ±0.15						

ARFIMA (0, d, 1) (Negligible sensitivity to initial conditions)

d	AR	MA	σ/var	AIC	SIC	MAX.LIK
2.36 (7.34)		1.03 (38.53)	0.19 (8.45)	515	508	259.5
conf. intvl ±0.10						

ARFIMA (1, d, 1) (Negligible sensitivity to initial conditions)

d	AR	MA	σ/var	AIC	SIC	MAX.LIK
2.39 (7.54)	0.08 (0.89)	1.02 (42.64)	0.19 (8.65)	513	504	259.8
conf. intvl ±0.10						

AFRIMA (2, d, 0) (Neglible sensitivity to initial conditions)

d	AR	MA	σ/var	AIC	SIC	MAX.LIK
2.48 (31.48) ±0.03	-0.45 (-6.39) 0.43 (6.31)		0.24 (9.1)	477	467	241.5

ARFIMA (2, d, 0) (No sensitivity to initial conditions)

d	AR	MA	σ/var	AIC	SIC	MAX.LIK
2.47 (11.80)	-0.22 (-3.07) 0.39		0.44 (9.34)	399	384	202.6
conf. intvl ±0.08	(5.55)					

Table 4. Chile

ARFIMA (0, d, 2) (Negligible sensitivity to initial conditions)

d	AR	MA	σ/var	AIC	SIC	MAX.LIK
2.43 (5.95)		0.86 (6.16) -0.16 (-1.20)	0.19 (8.63)	514	504	260.0
conf. intvl ±0.14						

ARFIMA (1, d, 2) (No sensitivity to initial conditions)

d	AR	MA	σ/var	AIC	SIC	MAX.LIK
2.46 (9.10)	-0.27 (-1.7)	0.57 (3.34) -0.46 (-3.64)	0.18 (8.57)	513	510	260.1
conf. intvl ±0.1						

ARFIMA (2, d, 2) (Moderate sensitivity to initial conditions)

d	AR	MA	σ/var	AIC	SIC	MAX.LIK
2.46 (10.34)	0.01 (0.04) 0.16 (1.73)	0.89 (3.12) -0.15 (-0.54)	0.18 (7.62)	513	498	261
conf. intvl ±0.90						

<u>Money Supply</u>

ARFIMA (1, d, 0) (Negligible sensitivity to initial conditions)

d	AR	MA	σ/var	AIC	SIC	MAX.LIK
2.31 (4.50)	-0.28 (-2.93)		0.51 (8.61)	376	370	190.2
conf. intvl ±0.13						

ARFIMA (0, d, 1) (Negligible sensitivity to initial conditions)

d	AR	MA	σ/var	AIC	SIC	MAX.LIK
2.2 (5.21)		1.0 (54.40)	0.35 (9.07)	436	430	220
conf. intvl ±0.07						

Table 4. Chile

ARFIMA (1, d, 1) (Negligible sensitivity to initial conditions)

d	AR	MA	σ/var	AIC	SIC	MAX.LIK
2.24 (4.65)	0.21 (2.35)	1.0 (61.32)	0.34 (8.84)	439	430	222.7
conf. intvl ±0.10						

ARFIMA (2, d, 1) (Negligible sensitivity to initial conditions)

d	AR	MA	σ/var	AIC	SIC	MAX.LIK
2.39 (6.70)	0.35 (3.94) 0.22 (2.46)	1.00 (50.77)	0.33 (9.23)	443	430	225
conf. intvl ±0.11						

ARFIMA (2, d, 2) (Little sensitivity to initial conditions)

d	AR	MA	σ/var	AIC	SIC	MAX.LIK
2.40 (4.0)	0.3 (0.78) 0.21 (2.75)	0.93 (2.00) -0.07 (-0.14)	0.32 (8.77)	441	425	225.7
conf. intvl ±0.20						

Table 5. Peru

Price Level

ARFIMA (1, d, 0) (Sensitivity to initial conditions)

d	AR	MA	σ/var	AIC	SIC	MAX.LIK
2.47 (12.79)	-0.51 (-7.64)		0.14 (11.02)	623	616	313.8
conf. intvl ±0.07						

ARFIMA (0, d, 1) (No sensitivity to initial conditions)

d	AR	MA	σ/var	AIC	SIC	MAX.LIK
2.48 (29.22)		1.00 (62.80)	0.09 (20.14)	702	695	353.0
conf. intvl ±0.03						

ARFIMA (1, d, 1) (Negligible sensitivity to initial condition)

d	AR	MA	σ/var	AIC	SIC	MAX.LIK
2.48 (27.16)	0.04 (0.57)	1.00 (62.80)	0.09 (10.38)	700	690	353
conf. intvl ±0.03						

ARFIMA (2, d, 0) (No sensitivity to initial conditions)

d	AR	MA	σ/var	AIC	SIC	MAX.LIK
2.49 (34.11)	-0.61 (-11.41)		0.13 (10.61)	635	624	
conf. intvl ±0.03						
	0.24 (3.99)					

Table 5. Peru

ARFIMA (0, d, 2) (Negligible sensitivity to initial conditions)

d	AR	MA	σ/var	AIC	SIC	MAX.LIK
2.49 (25.02)		0.96 (6.89)	0.09 (10.96)	700	690	353.1
conf. intvl ±0.04						
		-0.03 (-0.26)				

ARFIMA (1, d, 2) (Sensitivity to initial condition)

d	AR	MA	σ/var	AIC	SIC	MAX.LIK
2.49 (62.21)	0.19 (6.04)	0.15 (5.29)	0.09 (13.62)	698	684	353
conf. intvl ±0.01						
	1.15 (49.61)					

Money Supply

ARFIMA (1, d, 0) (No sensitivity to initial conditions)

d	AR	MA	σ/var	AIC	SIC	MAX.LIK
2.48 (20.96)	-0.21 (-3.21)		0.25 (11.17)	594	588	299.49
conf. intvl ±0.04						

ARFIMA (0, d, 1) (Little sensitivity to initial conditions)

d	AR	MA	σ/var	AIC	SIC	MAX.LIK
2.35 (9.37)		1.00 (80.11)	0.19 (10.7)	649	642	327.0
conf. intvl ±0.07						
		-0.95 (-13.43)				

Table 5. Peru

ARFIMA (2, d, 0) (Negligible sensitivity to initial conditions)

d	AR	MA	σ/var	AIC	SIC	MAX.LIK
2.49 (51.35)	-0.24 (-3.56)		0.23 (10.63)	606	595	306.1
conf. intvl ±0.02						
	0.24 (3.70)					

ARFIMA (2, d, 0) (Negligible sensitivity to initial conditions)

d	AR	MA	σ/var	AIC	SIC	MAX.LIK
2.47 (18.31)	-1.00 (-25.75)	-0.01 (-0.1)	0.19 (8.17)	635	621	
conf. intvl ±0.05						
		-0.95 (-13.43)				

ARFIMA (2, d, 2) (Sensitivity to initial conditions)

d	AR	MA	σ/var	AIC	SIC	MAX.LIK
2.48 (3.45)	1.16 (1.05)	1.82 (1.63)	0.16 (11.66)	679	662	344.8
conf. intvl ±0.03						
	0.43 (7.23)	0.82 (0.73)				

Table 6. Former Socialist Federal Republic of Yugoslavia

Money Supply

ARFIMA (1, d, 0) (No sensitivity to initial conditions)

d	AR	MA	σ/var	AIC	SIC	MAX.LIK
2.42 (6.00)	-0.5 (-5.21)		0.23 (9.85)	453	446	228.6
conf. intvl ±0.13						

ARFIMA (1 d, 1) (Negligible sensitivity to initial conditions)

d	AR	MA	σ/var	AIC	SIC	MAX.LIK
2.41 (5.94)	-0.11 (-1.05)	0.97 (33.4)	0.14 (9.72)	552	561	282.3
conf. intvl ±0.13						

ARFIMA (2, d, 0) (Negligible sensitivity to initial conditions)

d	AR	MA	σ/var	AIC	SIC	MAX.LIK
2.48 (28.64)	-0.6 (-8.57)		0.2 (9.67)	478	469	242.4
conf. intvl ±0.03						
	0.38 (5.73)					

ARFIMA (2, d, 1) (Little sensitivity to initial conditions)

d	AR	MA	σ/var	AIC	SIC	MAX.LIK
2.47 (14.04)	-0.45 (-6.22)	1.04 (37.36)	0.12 (8.95)	545	532	276.6
conf. intvl ±0.06						
	0.19 (2.54)					

Table 6. Former Socialist Federal Republic of Yugoslavia

ARFIMA (0, d, 2) (Sensitivity to initial conditions)

d	AR	MA	σ/var	AIC	SIC	MAX.LIK
2.38 (5.85)		1.26 (7.31)	0.13 (8.29)	543	533	244.6
conf. intvl ±0.12						
		0.24 (1.46)				

ARFIMA (1, d, 2) (Sensitivity to initial conditions)

d	AR	MA	σ/var	AIC	SIC	MAX.LIK
2.41 (7.67)	0.54 (4.58)	2.02 (16.86)	0.1 (5.87)	547	534	277.9
conf. intvl ±0.10						
		1.00 (8.36)				

Price Level

ARFIMA (1, d, 0) (Negligible sensitivity to initial conditions)

d	AR	MA	σ/var	AIC	SIC	MAX.LIK
2.43 (5.61)	-0.76 (-11.30)		0.09 (9.83)	612.4	606.0 308.2	
conf. intvl ±0.15						

ARFIMA (0, d, 1) (Sensitivity to initial conditions)

d	AR	MA	σ/var	AIC	SIC	MAX.LIK
2.49 (71.13)		0.98 (81.56)	0.07 (12.33)	650	643	327.2
conf. intvl ±0.01						

Table 6. Former Socialist Federal Republic of Yugoslavia

ARFIMA (2, d, 0) (Sensitivity to initial conditions)

d	AR	MA	σ/var	AIC	SIC	MAX.LIK
2.15 (1.5)	-0.70 (-0.94)		0.34 (18.45)	486	476	246.1
conf. intvl ±0.21						
	-0.29 (0.45)					

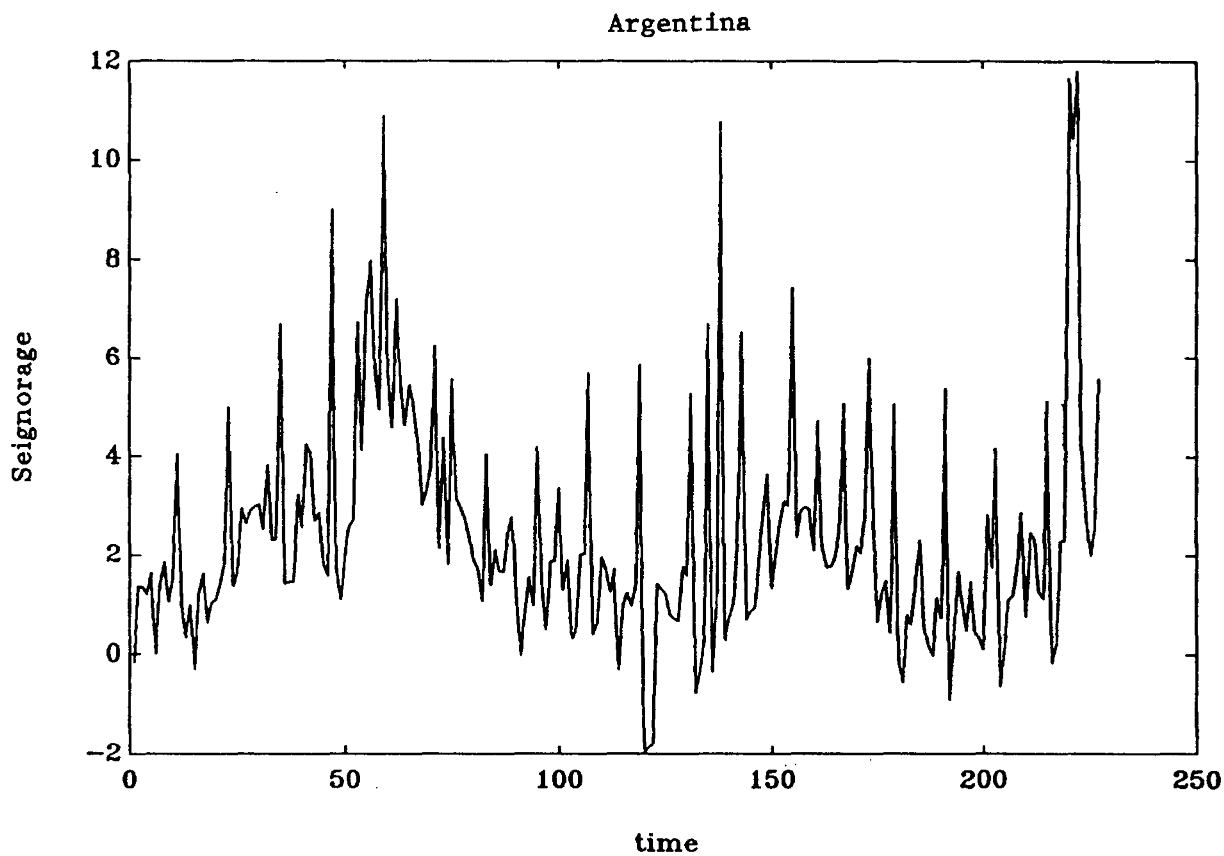
ARFIMA (0, d, 2) (Little sensitivity to initial conditions)

d	AR	MA	σ/var	AIC	SIC	MAX.LIK
2.48 (17.00)		1.29 (14.94)	0.06 (9.97)	666.6	656.9	277.9
conf. intvl ±0.05						
		0.31 (3.50)				

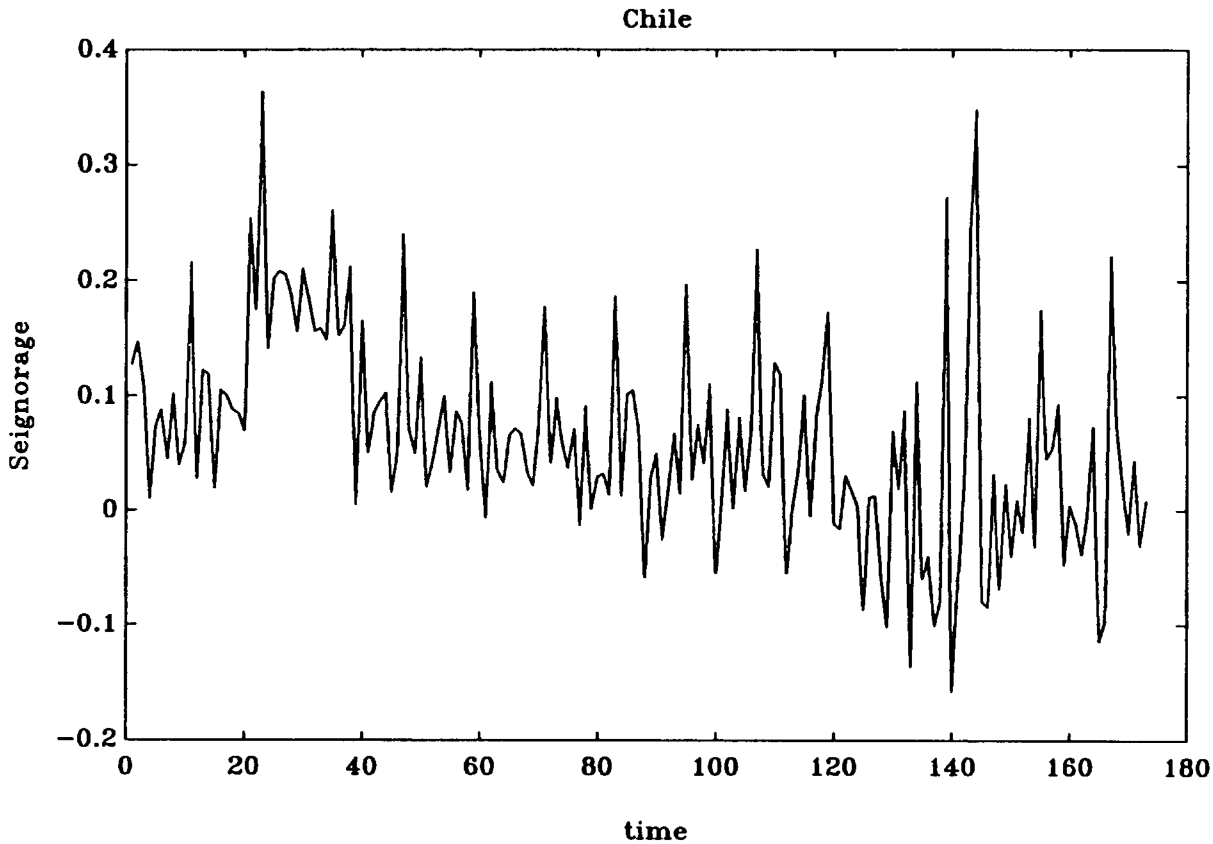
ARFIMA (1, d, 2) (Negligible sensitivity to initial conditions)

d	AR	MA	σ/var	AIC	SIC	MAX.LIK
2.15 (0.80)	-0.86 (11.30)	0.73 (5.10)	0.06 (9.65)	680	667	344.3
conf. intvl ±0.37						
		-0.23 (1.70)				

Graph 1



Graph 2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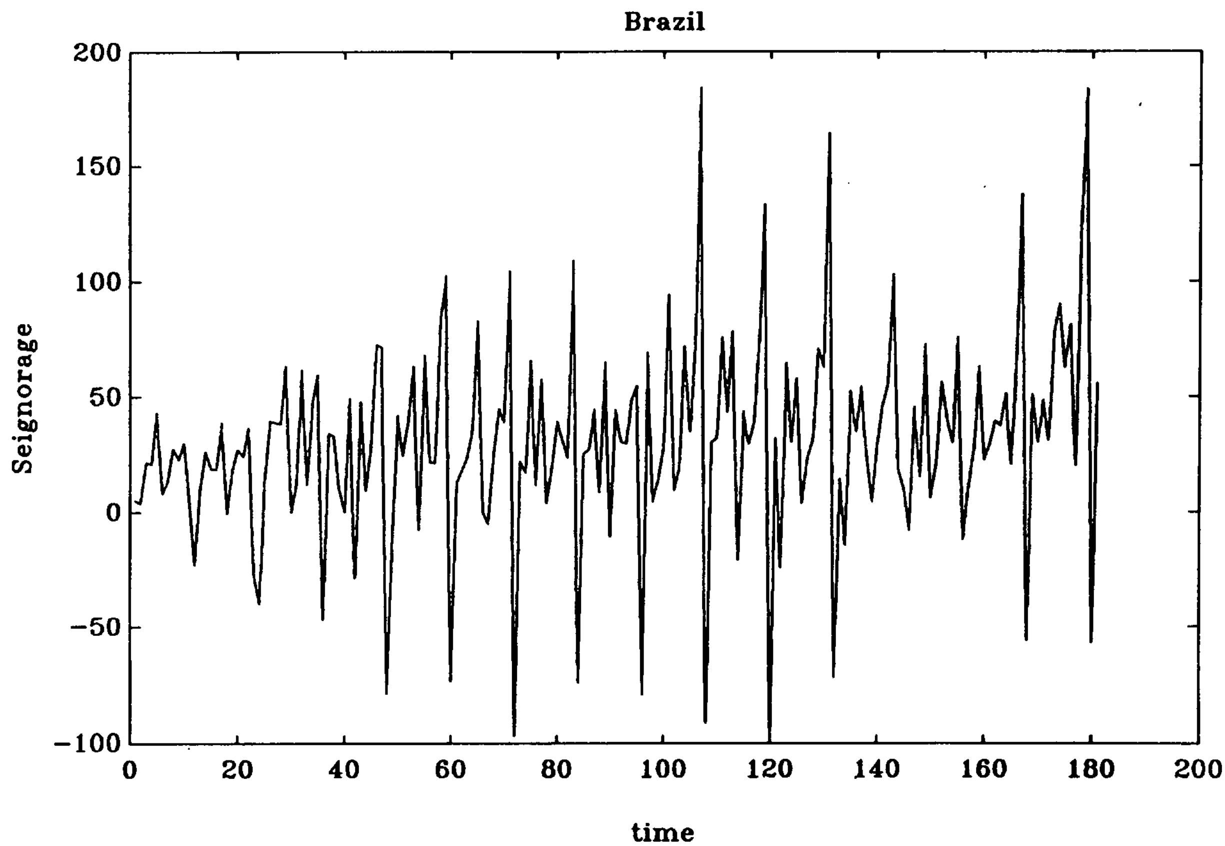
Graph 3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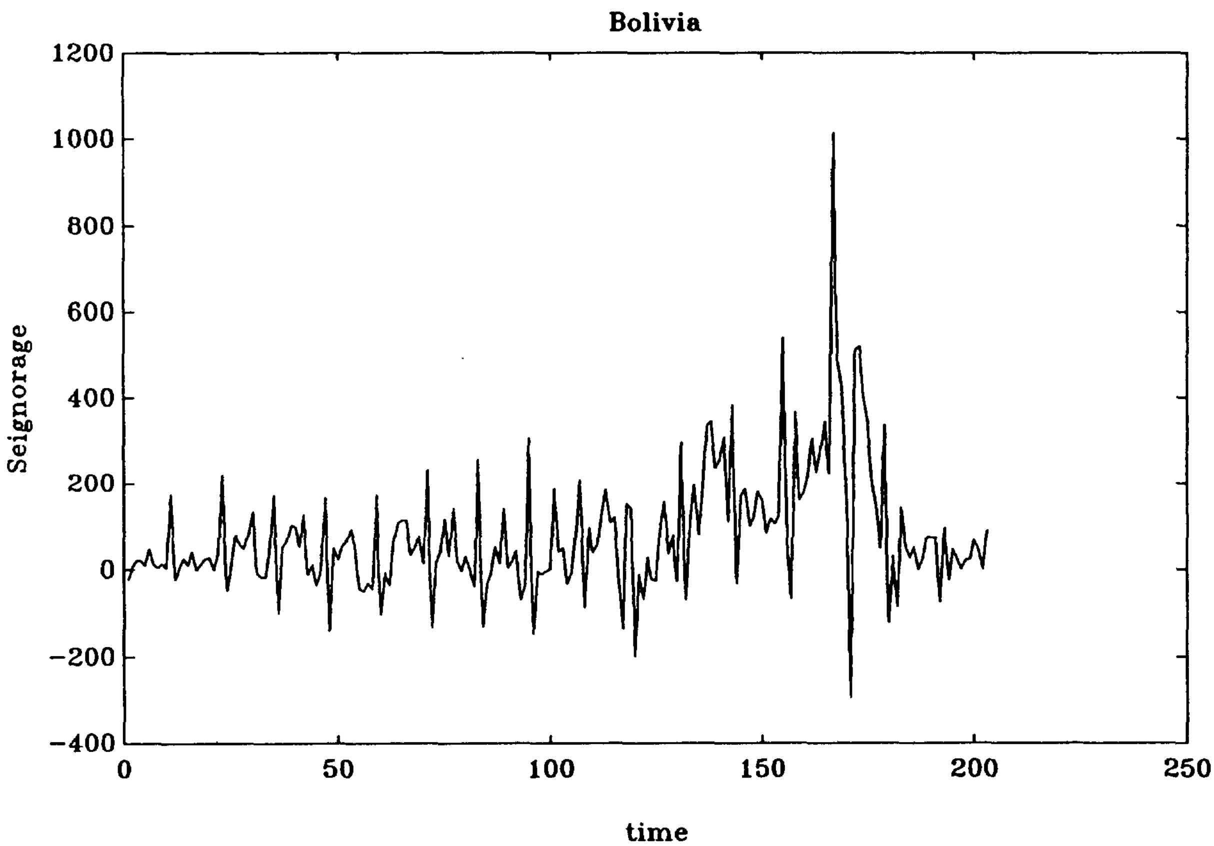
200

Graph 4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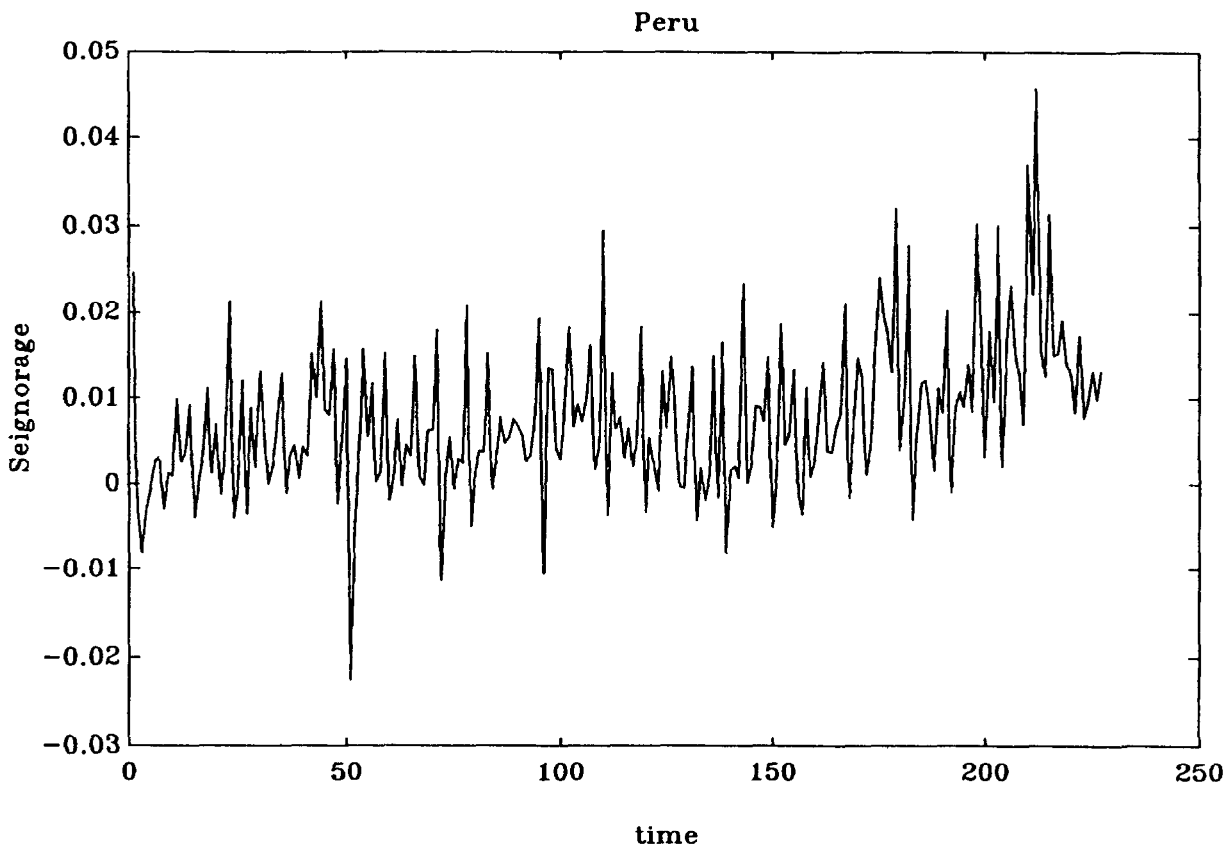
Graph 5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Graph 6

Former Socialist Federal Republic of Yugoslavia

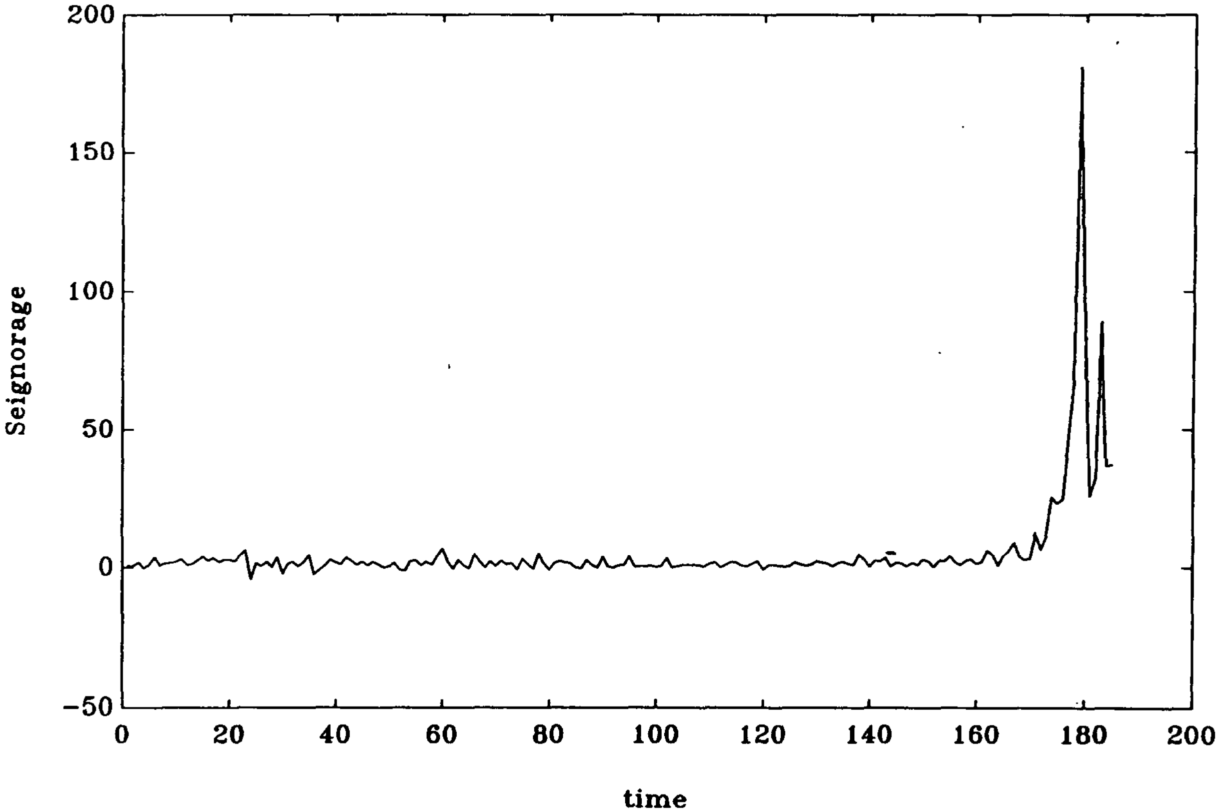
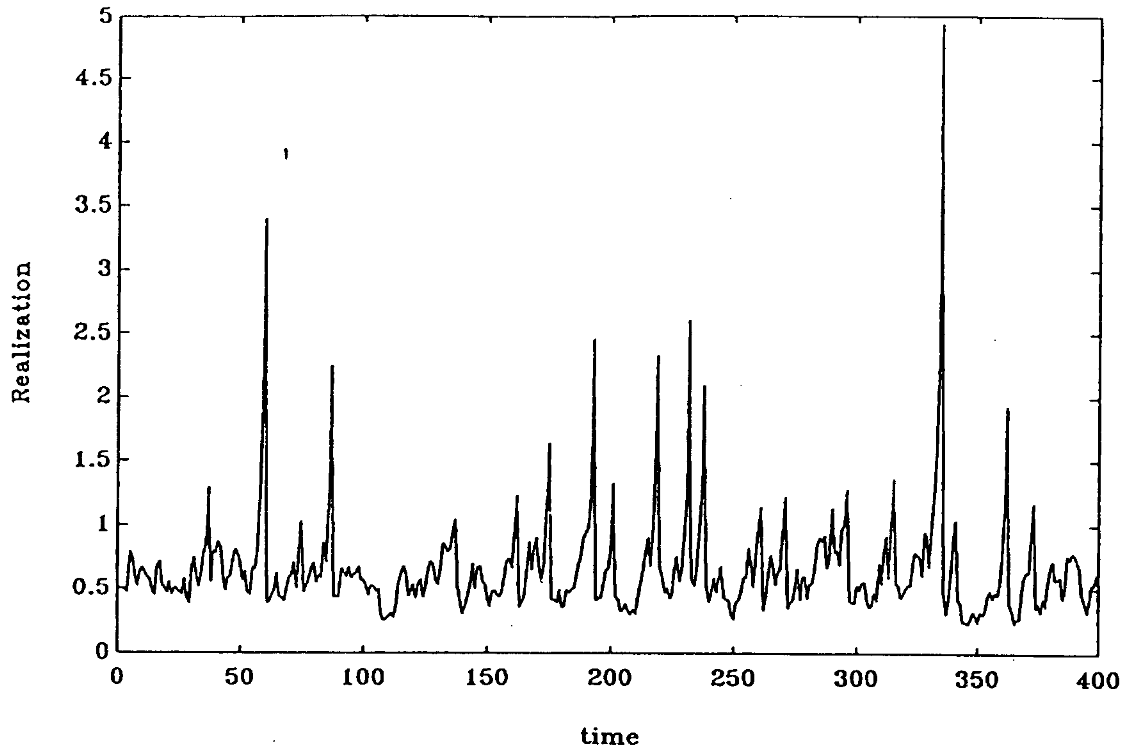
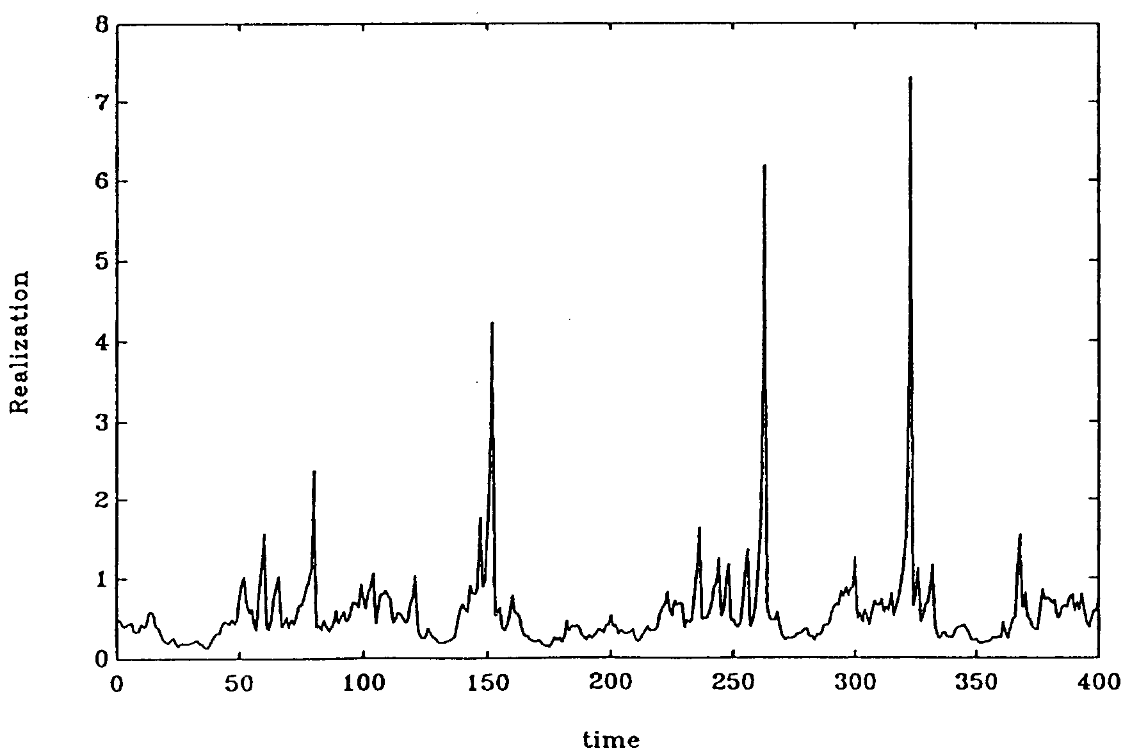


Chart 1. Simulated Bubbles





References

Blanchard, O.J., "Speculative Bubbles, Crashes, and Rational Expectations," Economics Letters, Vol. 3 (1979), pp. 387-389.

_____, and S. Fisher, Lectures in Macroeconomics (Cambridge: MIT Press, 1989).

Blackburn, K., and M. Sola, "Testing for Speculative Bubbles in Hyperinflations," London Business School Centre for Economic Forecasting, Discussion Paper No. 4-93 (1993).

Brock, W.A., "Money and Growth: The Case of Long-Run Perfect Foresight," International Economic Review, Vol. 15 (1987), pp. 750-777.

Broze, L., J. Janssen, and A. Szarfarz, "On Solutions of Linear Models with Rational Expectations," in J.P. Florens, M. Mouchart, J.P. Raoult, L. Simar (eds.), Alternative Approaches to Time Series Analysis-Proceedings of the 3rd Franco-Belgian Meeting of Statisticians (Brussels: Publications des Facultés Universitaires Saint-Louis, 1982).

Burmeister, E., and K.D. Wall, "Unobserved Rational Expectations and the German Hyperinflation with Endogenous Money Supply," International Economic Review, Vol. 28 (1987), pp. 15-32.

_____, R. Flood, P. Garber "On the Equivalence of Solutions in Rational Expectations Models", Journal of Economic Dynamics and Control, Vol. 5 1983), pp. 311-321.

Cagan, P., "The Monetary Dynamics of Hyperinflation," Studies in the Quantity Theory of Money, ed. by Milton Friedman (Chicago: University of Chicago Press, 1956), pp. 25-117.

Casella, A., "Testing for Rational Bubbles with Exogenous or Endogenous Fundamentals: the German Hyperinflation Once More," Journal of Monetary Economics, Vol. 24 (1989), pp. 109-122.

Campbell, J.Y., P. Perron, "Pitfalls and Opportunities: What Macroeconomists Should Know about Unit Roots" mimeo NBER Macroeconomic Conference (Cambridge: Massachusetts, March 1991).

Cochrane, J.H., "A Critique of the Application of Unit Root Tests" Journal of Economic Dynamics and Control, Vol. 15 (1991), pp. 275-284.

Dalhaus, R., "Efficient Parameter Estimation for Self-similar Processes," Annals of Statistics, Vol. 17 (1989), pp. 1749-1766.

Diba, B.T., and H.I. Grossman, "Rational Bubbles and the Price of Gold," NBER Working Paper No. 1300 (1984).

_____, "Rational Bubbles in Stock Prices," NBER Working Paper No. 1779 (1985).

_____, "Rational Inflationary Bubbles," Journal of Monetary Economics, Vol. 21 (1988), pp. 35-46.

Diebold, F.X., and G.D. Rudebush, "On the Power of Dickey-Fuller Tests Against Fractional Alternatives," Economics Letters, Vol. 35 (1991), pp. 155-160.

Evans, G.W., "Pitfalls in Testing for Explosive Bubbles in Asset Prices," American Economic Review, Vol. 81, No. 4 (1991), pp. 922-930.

_____, "Time Series Analysis of the German Hyperinflation," International Economic Review, Vol. 19 (1978), pp. 195-209.

Flood, R.P., and G.M. Garber (1980a), "Market Fundamentals Versus Speculative Bubbles: The First Tests," Journal of Political Economy, Vol. 88 (1980), pp. 745-770.

_____ (1980b), "An Economic Theory of Monetary Reform," Journal of Political Economy, Vol. 88 (1980), pp. 24-58.

_____, "A Model of Stochastic Process Switching," Econometrica, Vol. 51, No. 3 (1983), pp. 537-551.

_____, and L.O. Scott, "Multi-Country Tests for Price Level Bubbles," Journal of Economic Dynamics and Control, Vol. 8 (1984), pp. 329-334.

_____, and R.J. Hodrick, "Asset Price Volatility, Bubbles and Process Switching," NBER Working Paper No. 1867 (1986).

_____, and P. Kaplan, "An Evaluation of Recent Evidence on Stock Market Bubbles," NBER Working Paper No. 1971 (1986).

Fox, R., and M.S. Taqqu, "Large Sample Properties of Parameter Estimates for Strongly Dependent Stationary Gaussian Time Series," The Annals of Statistics, Vol. 14 (1986), pp. 517-532.

Froot, K.A., and M. Obstfeld, "Intrinsic Bubbles: The Case of Stock Prices," American Economic Review, Vol. 81 (1991), pp. 1189-1214.

Fuller, W.A., Introduction to Statistical Time Series (New York: Wiley, 1976).

Geweke, J., and H. Porter-Hudak, "The Estimation and Application of Long Memory Time Series Model," Journal of Time Series Analysis, Vol. 1 (1983), pp. 15-69.

Granger, C.W.J., "The Typical Spectral Shape of an Economic Variable," Econometrica, Vol. 34 (1966), pp. 151-161.

_____, "Long Memory Relationships and the Aggregation of Dynamic Models," Journal of Econometrics, Vol. 14 (1980), pp. 227-38.

_____, "Some Properties of Time Series Data and Their Use in Econometric Model Specification," Journal of Econometrics (1981), pp. 121-130.

_____, "Co-integrated Models and Error Correcting Models," UCSD Discussion Paper (1983).

Granger, C.W.J., and R. Joyeaux, "An Introduction to Long-Memory Time Series Models," Journal of Time Series Analysis, Vol. 4 (1980), pp. 221-238.

_____, and O. Morgenstern, "Spectral Analysis of the New York Stock Market Prices," The Random Character of Stock Market Prices (ed.), P. Cootner (Cambridge: MIT Press, 1964), pp. 162-88.

_____, and Weiss, A.A., "Time Series Analysis of Error Correction Models," Studies in Econometrics, Time Series and Multivariate Statistics (New York: Academic Press, 1983).

Gray, J.A., "Dynamic Instability in Rational Expectations Models: An Attempt to Clarify," International Economic Review, Vol. 25, No. 1 (1984), pp. 93-122.

Hamilton, J.D., "On Testing for Self-Fulfilling Speculative Price Bubbles," International Economic Review, Vol. 25 (1986), pp. 93-122.

_____, "A New Approach to the Economic Analysis of Non-Stationary Time Series and the Business Cycle," Econometrica, Vol. 57, No. 2 (1989), pp. 357-384.

_____, "Analysis of Time Series Subject to Changes in Regime," Journal of Econometrics, Vol. 45 (1990), pp. 39-70.

_____, and C. Whiteman, "The Observable Implications of Self-fulfilling Expectations," Journal of Monetary Economics, Vol. 16 (1985), pp. 353-74.

Hoskins, J.R.M., "Fractional Differencing," Biometrika, Vol. 68 (1981), pp. 165-176.

Janacek, G.J., "Determining the Degree of Differencing for Time Series via Log-Spectrum," Journal of Time Series Analysis, Vol. 3 (1982), pp. 177-183.

Jonas, A.B., "Persistent Memory Random Processes" (Cambridge: Harvard University, Department of Statistics), doctoral dissertation (1983).

Kashyap, R.L., and K.B. Eom, "Estimation in Long Memory Time Series Model," Journal of Time Series Analysis, Vol. 9 (1981), pp. 35-41.

Kingston G.H., "The Semi-log Portfolio Balance is Tenuous," Journal of Monetary Economics, Vol. 9 (1982), pp. 389-399.

Li, W.K., and A.E. McLeod, "Fractional Time Series Modelling," Biometrika, Vol. 73 (1986), pp. 217-221.

Mandelbrot, B.B., The Fractal Geometry of Nature (San Francisco: W.H. Freeman, 1982).

_____, and J. Van Ness, "Fractional Brownian Motion, Fractional Noises and Applications," SIAM Review, Vol. 10 (1968), pp. 422-437.

_____, and J.R. Wallis, "Noah, Joseph and Operational Hydrology," Water Resources Research, Vol. 4, No. 5 (1968), pp. 909-17.

Mandelbrot, B.B., and J.R. Wallis, "Computer Experiments with Fractional Gaussian Noises. Part 1, Part 2, and Part 3," Water Resources Research, Vol. 5, No. 1 (1969), pp. 228-266.

_____, and M.S. Taqqu, "Robust R/S Analysis of Long Run Serial Correlation," Proceedings of the 42nd Session of the International Statistical Institute, Manila, Bulletin of the I.S.I. 48, Vol. II (1979), pp. 69-104.

Matalas, N.C., and J. Wallis, "Small Sample Properties of H and K Estimators of the Hurst's Coefficient H," Water Resources Research, Vol. 7 (1970), pp. 1583-1594.

McLeod, A., and I. Hipel, "Preservation of the Rescaled Adjusted Range 1," Water Resources Research, Vol. 14 (1978), pp. 491-508.

Mohr, D.L. (1981a), "A Spectral Estimate of Hurst's Parameter" (Tulane University, Dept. of Mathematics), mimeographed (1981a).

Mussa, M., "Adaptive and Regressive Expectations in a Rational Model of the Inflationary Process," Journal of Monetary Economics, Vol. 1 (1975), pp. 423-442.

_____, "On the Inherent Stability of Rationally Adaptive Expectations," Journal of Monetary Economics, Vol. 4 (1978), pp. 307-314.

Muth, J.F., "Rational Expectations and the Theory of Price Movements," Econometrica, Vol. 29 (1961), pp. 315-35.

Phelps, E.S., Taylor J.B., "Stabilizing Powers of Monetary Policy under Rational Expectations," Journal of Political Economy, Vol. 85, pp. 163-190 (1977).

Phylaktis, K., and M.P. Taylor, "Money Demand, the Cagan Model and the Inflation Tax: Some Latin American Evidence," Review of Economics and Statistics, Vol. 75 (1993), pp. 32-37.

Sargent, T.J., "The Demand for Money During Hyperinflation Under Rational Expectations," International Economic Review, Vol. 18 (1977), pp. 59-82.

_____, Macroeconomic Theory, 2nd ed. (New York: Academic Press, 1987).

_____, Rational Expectations and Inflation (New York: Harper & Row, 1986).

_____, and N. Wallace, "Rational Expectations and the Dynamics of Hyperinflation," International Economic Review, Vol. 14 (1973a), pp. 328-350.

_____, and N. Wallace, "The Stability of Models of Money and Growth with Perfect Foresight," Econometrica, Vol. 41 (1973b), pp. 1043-1048.

_____, and N. Wallace, "Exploding Inflations" (mimeographed, Federal Reserve Bank of Minneapolis, 1984).

Scacciavillani, F., "Long Memory Processes in Speculative Markets," unpublished doctoral dissertation (Chicago: University of Chicago, 1993).

Sowell, F., "Modelling Long Run Behavior with the Fractional ARIMA Model," Journal of Monetary Economics, Vol. 29 (1992), pp. 277-302.

_____ (1989a), "The Deterministic Trend in Real GNP," mimeographed (Pittsburgh: Carnegie Mellon University, Graduate School of Industrial Administration, 1989).

_____ (1989b), "Maximum Likelihood Estimation of Fractionally Integrated Time Series Models," mimeographed (Pittsburgh: Carnegie Mellon University, Graduate School of Industrial Administration, 1989).

_____, "The Fractional Unit Root Distribution," Econometrica, Vol. 58 (1990), pp. 495-505.

_____, "Maximum Likelihood Estimation of Stationary Univariate Fractionally Integrated Time Series Models" (mimeographed, Pittsburgh: Carnegie Mellon University, Graduate School of Industrial Administration, 1991).

Taylor, M.P., "The Hyperinflation Model of Money Demand Revisited," Journal of Money, Credit and Banking, Vol. 23 (1991), pp. 327-349.

Taylor, J.B., "Staggered Wage Setting in a Macro Model," American Economic Review, Vol. 69 (1979), pp. 108-13.

Whiteman, C.H., Linear Rational Expectation Models: A User's Guide (Minneapolis: University of Minnesota Press, 1983).