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Debt, Deficits, and Age-Specific Mortality

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Abstract

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This paper develops an overlapping agents model with age-specific mortality rates. The analytical framework also nests Blanchard's (1985) "perpetual youth" model as a special, though perhaps not realistic, case. With age specific mortality rates, youth is "fleeting." Using standard hyperbolic functions, the model with fleeting youth is able to closely replicate the empirical relation between age and mortality. The comparative implications for deficit finance are also examined and age-specific mortality is shown to alter the non-Ricardian properties of the model.

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*"But at my back I always hear
Time's winged chariot hurrying near;
And yonder all before us lie
Deserts of vast eternity."*
—Andrew Marvell (1681)

I. INTRODUCTION

In the fifteen years since Blanchard's (1985) influential paper was published, numerous papers have been written utilizing his overlapping agents model to examine a wide range of economic issues. The framework—often referred to as the Blanchard-Yaari-Weil model—has proven to be a valuable analytical tool for researchers with its straightforward aggregation properties. But this desirable feature of the model also highlights one of its main drawbacks. The tractability of the model derives in part from the assumption that the probability of death is constant and the same for all agents regardless of their age. This constant hazard rate assumption has the peculiar feature that the planning horizon or expected time until death for an agent is the same at age 20 or 120. Blanchard himself recognized this limitation of the simple model that lacked any "life-cycle" dimension.

This paper derives a generalized version of Blanchard's celebrated model with age-specific mortality, incorporating the fact that mortality rates eventually rise with age. The model thus introduces a life-cycle dimension absent from the fixed mortality rate case. With a rising probability of death over an agent's lifetime, youth is no longer perpetual but "fleeting." Since this probability eventually reaches unity, the framework also places a well-defined age limit or upper bound on lifespans. This broader framework can also be shown to nest Blanchard's perpetual youth assumption as a special case.² By nesting both perpetual and fleeting youth, the framework can isolate the implications of mortality rates for economic behavior.

Specifically, it is a well known result that the mortality rate or finite horizons do not matter for debt nonneutrality in the Blanchard model. As shown by Weil (1987) and Buiter (1988), it is the birth rate not the death rate that underlies the failure of Ricardian Equivalence in the model. This result, however, is *special* to the perpetual youth assumption. Adding age-specific mortality, the distribution of mortality rates is shown to alter the real effects of public debt and identifies the circumstances under which finite horizons indeed matter.

The introduction of realistic age-specific mortality rates also allows the model to generate a more realistic age distribution. With perpetual youth, a significant right-tail to the distribution results, implying an overstatement of the number of elderly agents in the economy. This makes

²See Faruquee and Laxton (2000) for a discussion of the Blanchard model from both life-cycle and dynastic perspectives.

analysis of demographic and distributional issues problematic. With fleeting youth, having mortality rates the eventually increase corrects this distortion.

With the use of standard hyperbolic functions, the model with age-specific mortality is shown to be generally tractable. In addition to tractability, the hyperbolic specification is also shown to be sufficiently flexible to closely capture the empirical relation between age and mortality rates. Based on Gompertz's law, the parameters entering the mortality function are estimated. Using the estimated mortality function, the comparative implications of deficit finance are then simulated in the cases of perpetual and fleeting youth.

II. MODEL

To generalize the Blanchard model, we assume agents face a time-varying probability of death $p(s, t)$, where s is a generation index and t is the time index and their difference $t - s$ represents age. The number of survivors from a cohort born at time s at time t is given by:

$$n(s, t) = n(s, s) e^{-\int_s^t p(s, v) dv} = bN(s) e^{-\int_s^t p(s, v) dv}, \quad (1)$$

where b is the (fixed) birth rate—equal to the relative size of the newest cohort $n(s, s)$ as a proportion of the contemporaneous total population $N(s)$.³ From equation (1), the instantaneous change in a particular cohort's size is given by: $\dot{n}(s, t) = -p(s, t)n(s, t)$, where a dot represent the derivative with respect to time t .

Using the fact that the total population is the sum of all existing cohorts, the population growth rate is then given by:

$$\frac{\dot{N}(t)}{N(t)} = n - p, \quad (2)$$

where n is the rate of population growth and p is the *aggregate* mortality rate defined by the following adding-up condition:

$$p = \int_{-\infty}^t p(s, t) \frac{n(s, t)}{N(t)} ds. \quad (3)$$

³See Buiter (1988) for the extension of the Blanchard model to the case of population growth. Because the focus here is on the *adult* population, note that the "birth" rate denotes the arrival of new adults or workers (taken to be age 20).

This expression simply states that the overall death rate must equal the (weighted) average of age-specific death rates.⁴

The Blanchard model can be nested within these general conditions. The perpetual youth assumption is the special case with a static probability of death—i.e., where $p(s, t)$ is equal to a fixed p for all s and t . Furthermore, assuming a stationary population normalized to unity (i.e., $n = 0$ and $N = 1$) as we henceforth do, the birth rate b equals the death rate p . In that case, one exactly obtains Blanchard's survivors formula $n(s, t) = pe^{-p(t-s)}$ from (1).⁵

III. MORTALITY FUNCTION

If agents are taken to represent individuals, however, mortality rates should be age-dependent. Moreover, the mortality function should possess the following limit properties (for a given s):

$$\lim_{t \rightarrow s} p(s, t) = p_0 > 0, \quad (4)$$

$$\lim_{t \rightarrow \infty} p(s, t) = 1. \quad (5)$$

These Inada-type conditions require that the initial mortality rate be non-zero, and that the probability of death should eventually reach unity, reflecting a finite upper bound or age limit.⁶

The function should also be able to replicate the empirical relation between age and mortality. Following Blanchard (1985), "Gompertz's law" suggests that the death rate is roughly constant and near zero between age 20 and 40; beyond that, the mortality rate rises in the

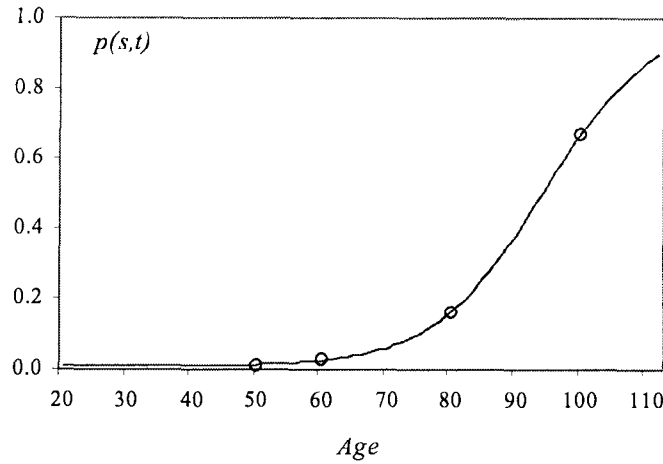
⁴Birth rates, aggregate death rates, and population growth are assumed constant throughout. See Faruquee (2001) for the case of time-varying birth and death rates and demographic dynamics. In the case of time-varying rates, individual mortality rates would be *cohort*-specific and not just *age*-specific.

⁵With no population growth, the following adding-up condition for $p(s, t)$ must be satisfied:

$$\frac{1}{p} \int_{-\infty}^t e^{-\int_s^t p(s, v) dv} ds.$$

⁶An alternative view of agents in the model, as discussed in Blanchard (1985), is that of dynastic households and where p is the probability that the family line ends. In that case, a common fixed hazard rate is more appealing.

Figure 1. Mortality Function and "Gomperty's Law"



following fashion: $p_{50}=1$ percent, $p_{60}=3$ percent, $p_{80}=16$ percent, $p_{100}=67$ percent. Figure 1 shows these data points (as open circles).

A functional form that can satisfy all these conditions and still remain fairly tractable is the hyperbolic tangent function:

$$p(s,t) = c_0 + c_1 \left[\tanh\left(\frac{t-s-k}{c_0}\right) + 1 \right] \quad (6)$$

where the parameters c_0 and c_1 are determined by the initial and terminal values of the mortality rate function—i.e., p_0 and p_1 , and k is a parameter representing the “mid-point” age—i.e., where $p(s,t) = c_0$.⁷ Blanchard's (1985) perpetual youth model obtains as the special case where $p_0 = p_1 = p$, implying that $c_0 = p$ and $c_1 = 0$. Otherwise, the case of fleeting youth has $0 < p_0 < p_1 = 1$, implying that $c_0 < c_1 = 1$.

In addition to satisfying conditions (4) and (5) in the case of fleeting youth, this functional form can also be shown to reproduce the empirical relation between mortality and age. Using non-linear least squares, structural time-series estimates of (6) shown fitting "Gomperty's Law" are as follows:

$$p(s,t) = 0.505 + 0.495 \tanh \left(\frac{t-s-74.2}{16.6} \right) \quad (7)$$

$$\bar{R}^2 = 0.999; D.W.=2.74.$$

⁷See appendix for a quick primer on hyperbolic functions.

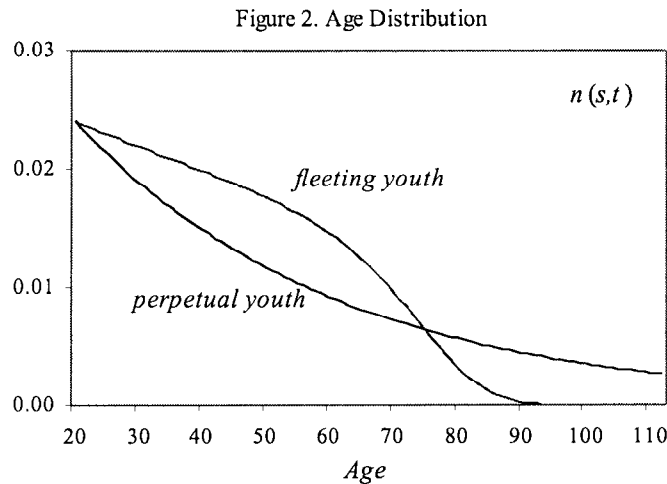
In the empirical specification, phase-shift and scaling parameters are both used to improve the fit. Corrected standard errors (not reported) indicate that all the parameters are significant at the 1 percent level. These non-linear estimates also impose the restriction that $\lim_{t \rightarrow \infty} p(s, t) = 1$ — i.e., $c_0 + c_1 = 1$.

Figure 1 plots the fit of the empirical equation (7) that reproduces "Gomperty's law". In a sense, the age-specific mortality rate model combines the features of Blanchard's probabilistic or uncertain lifetimes framework and a traditional life-cycle model with a known time of death (i.e., maximum age).

IV. AGE DISTRIBUTIONS

For a given aggregate mortality (and birth) rate, the distributional implications for the population of the two models can be compared. Specifically, the common death rate in the perpetual youth case is set equal to the weighted-average of individual mortality rates in the fleeting youth case.⁸ This allows the initial size of incoming cohorts to be the same in both settings, as well as the aggregate size of both populations.

The age distributions, however, are quite different as shown in figure 2. The vertical axis shows the relative size (in percent of the total population) of each age group. In the Blanchard model, a sizable tail in the age distribution exists, tending to overstate the number of old-aged persons in the economy. With age-specific mortality, a finite upper-bound or maximum age is quickly reached, truncating the right tail of the distribution.



⁸Using the adding-up condition and the estimated mortality function based on Gomperty's law, one can show that the implied aggregate mortality rate p is 2 ½ percent.

To further highlight these age-distribution differences, consider the following concrete example. With a world (adult) population of approximately 4 billion people, Blanchard's constant mortality rate would predict over 430 *million* persons or nearly 11 percent of the population to be age 110 years or older; for the same average death rate, the model with age-specific mortality rates would predict less than 5 individuals.

Consumer's problem

With log utility and assuming that utility is zero after the agent dies, the consumer's problem can be formulated as maximizing the following expected utility function:

$$E_t \int_t^{\infty} \ln c(s, v) e^{-\theta(v-t)} dv = \int_t^{\infty} \ln c(s, v) e^{-\int_t^v \{\theta + p(s, z)\} dz} dv, \quad (8)$$

where θ is the rate of time preference. The mortality function $p(s, t)$ introduces an additional discounting term that is now age-specific, based on the dynamic probability that the agent dies in some future period.

The treatment of the annuities or insurance market becomes more complicated in the case of age-specific mortality rates. Specifically, the zero profit condition for firms assumed in Blanchard (1985) and drawing on Yaari (1965) could be interpreted in two distinct ways here. If firms cannot use the fact that hazard rates are age-specific, they could still balance their books based on population averages for mortality rates.⁹ If insurance firms could discriminate (by age), however, they could make net payments $p(s, t)w(s, t)$ to individual agents in exchange for all their wealth contingent upon death. This would allow the insurance company to balance its payments and receipts *by* age group. This latter scheme is adopted here:¹⁰

$$\dot{w}(s, t) = [r(t) - p(s, t)]w(s, t) - y(s, t) + \tau(s, t) - c(s, t); \quad (9)$$

where r is the interest rate, $y - \tau$ is disposable labor income and c is consumption.

Solving the consumer's problem yields the following optimal solution for individual consumption :

$$c(s, t) = \phi(s, t)[w(s, t) + h(s, t)] \quad (10)$$

⁹Firms could pay all age groups at the same average rate, paying $\tilde{p}w(s, t)$ in return for the estate at the time of death, provided that the common payout rate \tilde{p} equaled a wealth-weighted average of individual mortality rates.

¹⁰We operate under the assumption of price discrimination to further distinguish the model from the original assumptions used in Blanchard (1985). Derivations of the model under the common pricing rule could also be derived (see previous footnote).

where $\phi(s, t)$ is the marginal propensity to consume out of wealth, and $h(s, t)$ is defined in the conventional way as human wealth or the present value of the future stream of disposable labor income.^{11, 12} The inverse of marginal propensity to consume $\phi^{-1}(s, t) \equiv \phi(s, t)^{-1}$ evolves according to the following equation:

$$\dot{\phi}^{-1}(s, t) = [\theta - p(s, t)]\phi^{-1}(s, t) - 1. \quad (11)$$

In the case of perpetual youth, the marginal propensity to consume would be constant—i.e., $\dot{\phi}^{-1}(s, t) = 0$, and identical for all agents: $\theta = p$.

The dynamics for aggregate consumption can be written generally as:

$$\dot{C}(t) = pc(t, t) - (r(t) - \theta)C(t) - \int_{-\infty}^t p(s, t)c(s, t)n(s, t)ds, \quad (12)$$

where uppercase variables denote economy-wide aggregates.¹³ Equation (12) shows that the evolution of total consumption depends on the additional consumption of newly arriving agents, the sum of the consumption changes of existing agents, and subtracting out the consumption of those

¹¹Solving the maximization problem, one can show that with log utility:

$$\phi(s, t)^{-1} = \int_t^{\infty} e^{-\int_t^v \{\theta + p(s, z)\} dz} dv.$$

This expression can be shown to have a closed-form solution given the functional form for $p(s, t)$. If $p(s, t) = p$, then $\phi(s, t)$ is constant ($\theta = p$) as in Blanchard (1985).

¹²Integrating forward the agent's dynamic budget constraint (9), we have $h(s, t)$ given by:

$$h(s, t) = \int_t^{\infty} \{y(s, v) - \tau(s, v)\} e^{-\int_t^v \{r(z) + p(s, z)\} dz} dv.$$

¹³Note that aggregate variables denoted with uppercase are defined as follows:

$$C(t) \equiv \int_{-\infty}^t c(s, t)n(s, t)ds.$$

Differentiating this expression and using the Keynes-Ramsey rule for optimal consumption, yields equation (12).

agents that die each period.¹⁴ In the simple perpetual youth case, this last term would be given by $pC(t)$.

The dynamics of aggregate financial and human wealth are given by:

$$\dot{W}(t) = r(t)W(t) - Y(t) - T(t) - C(t) \quad (13)$$

$$\dot{H}(t) = \phi h(t, t) - r(t)H(t) - [Y(t) - T(t)] \quad (14)$$

The dynamics of financial wealth are identical to the Blanchard model, reflecting a budget constraint for the economy as a whole. The dynamics of human wealth, however, are different and now include the marginal contribution of newly arriving agents' human wealth to the economy-wide aggregate. If disposable labor income is not age-specific—i.e., $y(s, t) = \tau(s, t) - Y(t) - T(t)$ and youth was perpetual, individual human wealth would *not* be age-specific—i.e., $h(t, t) = H(t)$; then, equation (14) would reduce to the Blanchard case.¹⁵

Based on individual consumption behavior and the fact that the marginal propensity to consume is the same across age groups under perpetual youth, the solution for aggregate consumption in the Blanchard case is well known:

$$C(t) = (\theta + p)[W(t) + H(t)]. \quad (15)$$

Differentiating this expression yields the dynamics for consumption:

$$\dot{C}(t) = (r(t) - \theta - p)C(t) + p(\theta + p)H(t). \quad (16)$$

In the age-specific mortality case, obtaining a closed-form solution for aggregate consumption is somewhat more complicated. Using the mean-value theorem, however, an analytical solution can be expressed as follows:

$$C(t) = c(\xi, t)n(\xi, t)\Lambda, \quad (17)$$

¹⁴Note that since agents are born without wealth—i.e., $w(t, t) = 0$, initial consumption is given by $c(t, t) = \phi(t, t)h(t, t)$.

¹⁵The case of life-cycle income is described in the appendix.

where ξ is the index representing the mean-value cohort in total consumption, and Λ is related to the finite range of ages in the population with fleeting youth.¹⁶ Using the mean-value theorem again and equation (12), the law of motion for consumption is given by:

$$\dot{C}(t) = (r(t) - T)C(t) - p(\zeta, t)c(\zeta, t)n(\zeta, t) + p\phi(t, t)h(t, t), \quad (18)$$

where ζ is the mean-value cohort for the total consumption of agents who die, whereas ξ is the mean-value cohort for the consumption of those still alive.¹⁷

V. STEADY-STATE CONSUMPTION BEHAVIOR

Using the behavioral equations derived above, the consumption and saving behavior under perpetual and fleeting youth can be compared. The model with fleeting is solved using the fact that an individual's steady-state time profile for key behavioral measures can also be used to describe the cross-sectional pattern across agents of different ages. Specifically, at very old ages—i.e., as $p(s, t) \rightarrow 1$, the marginal propensity to consume converges to a well-defined upper bound:

$l(s, t) \rightarrow \theta + 1$; human wealth, meanwhile, falls to following lower bound:

$h(s, t) \rightarrow (\bar{Y} - \bar{T})/(\bar{r} + 1)$, where bars denote steady-state values. Using these terminal conditions and the laws of motion for ϕ and h , one can then solve backwards for these values for younger agents.¹⁸

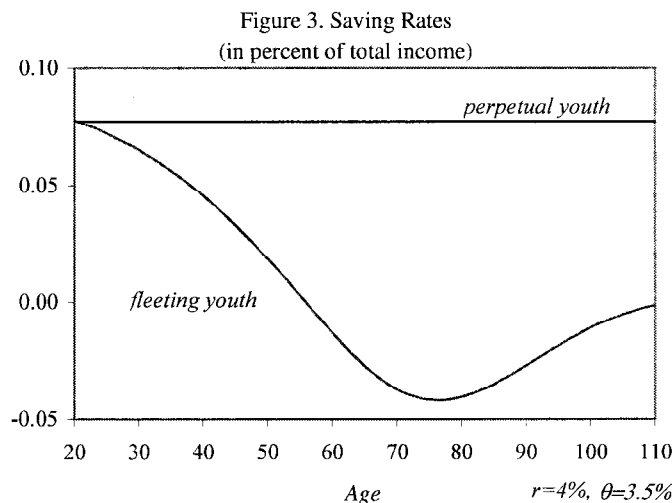
Assuming the same rate of time preference and interest rate, individual saving rates in steady state implied by optimal consumption plans are shown in figure 3. The saving rate is defined as the rate of change in financial wealth. With perpetual youth, individual saving rates are always positive and the same across all age groups. Again, in the absence of any life-cycle features, this result is not surprising. Agents face a fixed planning horizon independent of their age and choose to save at a

¹⁶Using the mean value theorem, we can then write:

$$c(\xi, t)n(\xi, t) = \int_{-\Lambda}^{\xi} c(s, t)n(s, t)ds, \text{ for } \xi \in (t - \Lambda, t).$$

¹⁷It should be noted that the Λ terms in (17) and (18) need *not* be the same as they refer to the relevant age ranges for two different measures: $c(s, t)n(s, t)$ and $p(s, t)c(s, t)n(s, t)$. Because $p(s, t) \in (0, 1)$, it follows that the age range is generally smaller in the second instance.

¹⁸As a consistency check, one can also solve the integral calculus problem directly to obtain (say) $\phi(s, t)$ directly. With $\theta = 3.5\%$, the model suggests a marginal propensity to consume of 5% for young agents (under age 30), rising to 10% by age 60, and 25% by age 80. These direct calculations are indeed consistent with the saving rates in figure 3. [Details of the integration solution in *Mathematica*[®] are available from the author upon request.]



fixed rate over time. This allows steady wealth accumulation and rising consumption over an individual's lifetime.¹⁹

With fleeting youth, a clear life-cycle pattern of saving emerges. Across age groups, young and middle-age agents tend to be net savers, while older agents facing shorter time horizons (higher discount rates) tend to dissave to some extent. Eventually, saving rates settle down near zero.²⁰

Interestingly, initial saving rates across models are the same in figure 3, though this need *not* always be the case. The probability of death and, hence, the effective discount factor are lower initially with the fleeting youth. Consequently, the marginal propensity to consume is comparatively lower initially, but human wealth is higher for the same reason. For the interest rate and time preference assumed here, these opposing effects just cancel, leaving initial consumption rates to be essentially the same in the two cases.

These comparative implications should be taken as only partial results since the equilibrium interest rate will generally *not* be the same across the two models. The fact that saving rates are uniformly lower for a given interest rate suggests less wealth and capital accumulation, and thus a higher long-run equilibrium interest rate in a model with fleeting youth.

¹⁹Recall that although individual consumption is continuously rising—with $\bar{r} > \theta$, aggregate consumption growth is zero in steady state through population turnover as discussed in Blanchard (1985).

²⁰These values could imply small positive saving rates near the end of life, depending on the value of $\bar{r} - \theta$.

VI. STEADY-STATE EFFECTS OF GOVERNMENT DEBT

From a policy perspective, an important question is whether the introduction of age-specific mortality rates affects the impact of public debt. Changes in debt—i.e., the timing of taxes—have real effects in the standard model because public and private discount rates differ. Thus, national saving could fall in response to an increase in the budget deficit.

To examine the debt implications of age-specific mortality, we first introduce the government sector in the standard way:

$$\dot{D}(t) = G(t) - T(t) + r(t)D(t), \quad (19)$$

where D is the stock of outstanding government debt, G is public expenditures and T is (lump-sum) taxes. Financial wealth held by the private sector consists of government bonds and the capital stock: $W = D + K$.

For convenience, government expenditures are assumed to be fixed at zero. In the initial steady state, taxes and debt are also assumed to be zero (i.e., $\bar{D} = \bar{T} = 0$). Otherwise, from the government's budget constraint, the following long-run relation between taxes and debt holds:

$$\frac{\bar{T}}{\bar{D}} = \bar{r}. \quad (20)$$

The initial steady state in the perpetual youth case is one where the capital stock is equal to unity.²¹ From the economy's budget constraint, this implies that steady state consumption is also unity. In the case of fleeting youth, because saving is uniformly lower for a given interest rate, there is less capital accumulation and a higher equilibrium real interest rate than in the perpetual youth case. Consequently, the capital stock and consumption are less than unity with fleeting youth, given identical taste and technology parameters.

Now consider a unit shock to government debt. From equation (20), taxes in the new steady state will be positive, equal to the new equilibrium interest rate. The interest rate will also be higher as the increase in public debt crowds out to some extent the capital stock. Because public and private discount rates differ, a shift in the timing of taxes (i.e., deficit finance) will affect national saving and investment as agents do not fully internalize the future tax implications of higher debt that follow

²¹With a standard Cobb-Douglas production function, a unit capital (and labor) stock requires that capital share be equal to $\theta + p(\theta + p)$, which we assume for convenience. In the perpetual youth case, the initial steady state real interest rate will also equal $\theta + p(\theta + p)$. In the fleeting youth case, however, the equilibrium interest rate will be higher than this value given lower saving rates and smaller capital stock.

from the government's intertemporal budget constraint. The extent to which debt has real effects in the two models is shown in table 1.

As seen in the table, the real effects of government debt are significantly higher in the case of fleeting youth.²² The intuition for this result can be seen from equation (17). The mean-value cohort in consumption has a higher discount rate (and higher mortality rate) than the population-wide average. Consequently, the wedge between public discount rate and private discount rate in consumption is larger. A larger wedge between discount rates induces a larger departure from debt neutrality or Ricardian equivalence.

Table 1. Steady-State Effects of Government Debt
(change in capital stock from initial values)

Model	$\theta = 1\%$	$\theta = 3.5\%$	$\theta = 10\%$
<i>Perpetual Youth</i>	-7.4%	-4.2%	-3.5%
<i>Fleeting Youth</i>	-20.1%	-10.1%	-7.0%

Apart from these quantitative results, it is useful to compare these basic findings to other studies. In a paper extending the Blanchard model to include population growth (i.e., when birth and death rates could differ), Buiter (1988) showed that birth rates *not* death rates were the reason behind debt non-neutrality in the case of perpetual youth.²³ The intuition is as follows. New taxpayers arriving in the future allowed current individuals—who share no intergenerational connection—to enjoy the benefits of lower taxes today without bearing all of the future tax burden. The *level* of the mortality rate, however, did not matter. In other words, finite horizons were not, in any way, the source of the failure of Ricardian equivalence in the Blanchard model.

This analysis shows, however, that the *distribution* of mortality rates does matter for debt non-neutrality. In particular, for a given average death rate, age-specific mortality according to Gompertz's law tends to increase the crowding-out effects of government debt. The reason is that the effective private discount rate—represented by the mean-value cohort—in consumption is higher with fleeting youth. Seen from another perspective, the *consumption*-weighted mortality rate is higher than the population-weighted mortality rate. Older agents with higher than average discount

²²The small effect of government debt in the perpetual youth model is a well-known result; see Evans (1990). Adding age-earnings profiles, Faruquee, Laxton and Symansky (1997) and Faruquee and Laxton (2000) show, however, that the departures from Ricardian equivalence can be much larger. See appendix for the case of life-cycle income with fleeting youth.

²³Weil (1997) derives a similar result in an infinite horizon model with population growth.

rates tend to comprise a larger share of total consumption. Hence, finite horizons matter in this context as Blanchard had originally posited, by increasing the wedge between public and private discount rates. This augments the real effects of public debt, beyond the impact of a positive birth rate. In Blanchard's special case of perpetual youth, these distributional implications regarding discount rates vanish, and, as shown by Buiter (1988), the role of finite horizons is eliminated.²⁴

VII. CONCLUSIONS

This paper extends Blanchard's (1985) overlapping agents model to include age-specific mortality rates. Using hyperbolic functions to introduce mortality rates that replicate Gompertz's law, the model with fleeting youth produces several areas of departure from the original model. The introduction of age-specific mortality allows a more direct interpretation of agents as individuals, facing realistic probabilities of death at different ages. This also produces a demographic age structure more in line with reality. Moreover, fleeting youth introduces a form of heterogeneity across agents and life-cycle implications that were absent from the simpler model. Finally, the change (from a flat distribution) in mortality rates increases the real effects of government debt, as finite horizons now matter. Older agents with higher than average discount rates tend to comprise a larger share of total consumption. Age-specific mortality thus increases the wedge between the public discount rate and the effective private discount rate, inducing a larger departure from Ricardian equivalence.

²⁴It is straightforward to prove that the consumption-weighted mortality rate is *identical* to the overall mortality rate in the perpetual youth case: $C(t)^{-1} \int_{-\infty}^{\infty} pc(s,t)n(s,t)ds = p = \int_{-\infty}^{\infty} pn(s,t)ds$.

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Hyperbolic Functions

The appendix presents a quick primer on the hyperbolic functions used to derive analytically tractable solutions. The basic hyperbolic sine, cosine, and tangent functions are defined as follows:

$$\sinh(x) = \frac{1}{2}(e^x - e^{-x}), \quad (A1)$$

$$\cosh(x) = \frac{1}{2}(e^x + e^{-x}), \quad (A2)$$

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)}. \quad (A3)$$

These functions have straightforward integrals and derivatives and possess properties that, in many instances, are broadly similar to their counterpart trigonometric functions. Some useful results are as follows:

$$\cosh^2(x) - \sinh^2(x) = 1, \quad (A4)$$

$$\int \cosh(x) dx = \sinh(x), \quad (A5)$$

$$\int \sinh(x) dx = \cosh(x), \quad (A6)$$

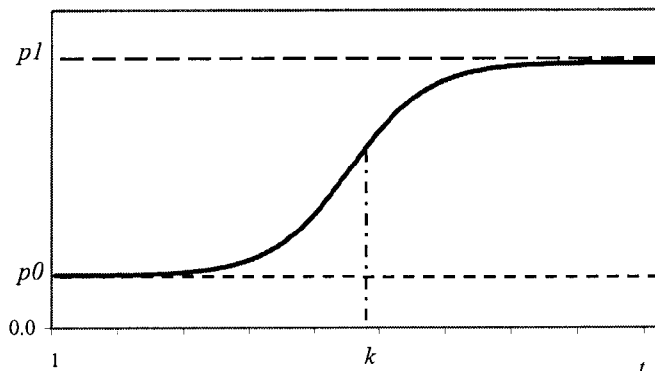
$$\int \tanh(x) dx = \log[\cosh(x)]. \quad (A7)$$

A more elaborate parameterization of the hyperbolic tangent function can be written as follows:

$$p(t) = c_0 + c_1 \left[\tanh \left(\frac{t - k}{a} \right) \right]; \quad c_0 = \frac{p_0 + p_1}{2}, \quad c_1 = \frac{p_1 - p_0}{2}. \quad (A8)$$

In equation (A8), $p_0 = c_0 - c_1$ and $p_1 = c_0 + c_1$ represent the limit values of the function as $t \rightarrow -\infty$ and $t \rightarrow \infty$ respectively; a is a scaling parameter and k is a phase-shift parameter, marking the point in time when the value of the function is exactly half-way between the limit values—i.e., $p(k) = c_0$. Figure A1 depicts the "step" function in (A8) for $a = 1, k > 0$.

Figure A1. Hyperbolic Tangent Function



Life-Cycle Income

To further incorporate life-cycle aspects to the analysis, the framework can be extended to the case where labor income is also age-specific, rising with age and experience when agents are young, before eventually declining with retirement when agents are old.

To introduce a hump-shaped earnings profile over an individual's lifetime, we assume that the income $y(s,t)$ accruing to an individual from generation s at time t can be expressed in terms of age-dependent weights on aggregate labor income $Y(t)$, to allow for aggregation, equal to the sum of two exponential functions:²⁵

$$y(s,t) = \left[a_1 e^{-\alpha_1(t-s)} + a_2 e^{-\alpha_2(t-s)} \right] Y(t); \quad a_1 > 0, a_2 < 0, \alpha_2, \alpha_1 > 0. \quad (\text{A9})$$

The first exponential can be interpreted as the gradually declining endowment of labor (i.e., gradual retirement) which is inelastically supplied. The second exponential can be interpreted as the relative productivity and wage gains from experience (or smaller costs of inexperience) with increasing age.²⁶

With proportional labor income taxes, individual human wealth with age-earnings profiles can be summarized as follows:²⁷

$$h(s,t) = h_1(s,t) + h_2(s,t) \quad (\text{A10})$$

$$h_1(s,t) = a_1 e^{-\alpha_1(t-s)} \int_t^\infty \{(1-\tau)Y(v)\} e^{-\int_t^v \{\alpha_1 + r(z) + p(s,z)\} dz} dv \quad (\text{A11})$$

$$h_2(s,t) = a_2 e^{-\alpha_2(t-s)} \int_t^\infty \{(1-\tau)Y(v)\} e^{-\int_t^v \{\alpha_2 + r(z) + p(s,z)\} dz} dv \quad (\text{A12})$$

Using these equations, the evolution of individual human wealth for the *newest generation* at each point in time is given by:

$$\dot{h}(t,t) = \dot{h}_1(t,t) + \dot{h}_2(t,t) \quad (\text{A13})$$

²⁵ The parameters in the weighting function are assumed to be always non-negative and initially increasing; by an adding up constraint, we also require that

$$\frac{1}{p} = \int_{-\infty}^t \left\{ a_1 e^{-\alpha_1(t-s)} + a_2 e^{-\alpha_2(t-s)} \right\} e^{-\int_s^t p(s,v) dv} ds.$$

²⁶ See Faruquee and Laxton (2000) for empirical estimates of hump-shaped age-earnings profiles for the United States; see Faruquee (2000) for estimates of the age-earnings profile for Japan.

²⁷ Blanchard (1985) examines the case of monotonically declining income profiles.

$$\dot{h}_1(t, t) = [r(t) + p(t, t) + \alpha_1]h_1(t, t) - a_1(1 - \tau)Y(t) \quad (\text{A14})$$

$$\dot{h}_2(t, t) = [r(t) + p(t, t) + \alpha_2]h_2(t, t) - a_2(1 - \tau)Y(t) \quad (\text{A15})$$

The human capital of new agents then enters the aggregate equation as discussed in the text:

$$\dot{H}(t) = ph(t, t) + r(t)H(t) - (1 - \tau)Y(t) . \quad (\text{A16})$$

In terms of their economic implications, life-cycle income profiles tend to *augment* the effects of government debt on the real economy. Specifically, age-earnings profiles further increase the wedge between public and private discount rates, seen by the α terms in equations (A11) and (A12), beyond the effects of a positive death and birth rate ($p > 0$). In other words, the policy choice between tax financing versus deficit financing (i.e., the timing of taxes) will have larger consequences for national consumption and saving, if agents perceive that the prospective tax burden falls partly on future generations who also have relatively higher taxable income.²⁸

²⁸ See Faruquee, Laxton and Symansky (1998) and Faruquee and Laxton (2000) for an analysis of hump-shaped age-earnings profiles in the Blanchard model.