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The Equilibrium Distributions of Value for Risky Stocks and Bonds

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Abstract

Within a unified theory for stocks and corporate bonds, based on dynamic optimization by investors, this paper derives analytical expressions for the momentary distributions of expected price, respectively known to approximate lognormal with systematic deviations (high peak, fat tail) and double exponential (for credit risk). Market equilibrium is regarded as a dynamic equilibrium characterized by a time-invariant probability distribution over microfinancial states, marginal redistributions of portfolios are regarded as indistinguishable, and real and fiat assets are regarded as essentially distinct. The formalism provides a basis for decomposing value changes by market fundamentals, investor sentiment, and investor acquisition of securities.

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I. INTRODUCTION

Almost always, it seems, capital assets occur as one of two kinds: either **real** or **fiat**. Real assets are characterized by a limited rate of increase (possibly zero or negative) in amounts outstanding, whereas fiat assets have a potentially unlimited rate of increase. Real assets (e.g. physical assets) are subject to varying marginal product, whereas fiat assets (e.g. a contract with an obligor) are subject to constant marginal product (assuming that investor creditworthiness ceilings restrict the aggregate obligations). In some sense, therefore, real assets underlying a financial asset do not occupy identical states within a financial system, whereas fiat assets do.

We shall crystallize these considerations in the form of an axiom of **allowed residency** of identical states, namely, for real assets, single residency at most, and for fiat assets, unlimited residency. For the sake of making a concrete application, we shall consider stocks as an exemplar of a financial asset supported by real assets, and corporate bonds of one supported by fiat assets; a unified theory for the two will be developed on the basis of optimizing behaviour in frictionless markets.

The financial system that we shall consider consists of a set of assets of fixed types, comprising a stock market and a bond market (including a money market deposit), in equilibrium with other asset markets. In order to abstract from issues of term structure, however, bonds are considered to be of indefinite maturity and subject only to credit risk. Investors (the treatment of whom will be largely implicit) optimally manage their portfolios through time, exercising short-run rational expectations such that prices evolve deterministically. However, because portfolio cashflows are discontinuous (for instance, discrete dividend or interest payments), they trade in finite (as distinct from marginal) quantities at discrete points in time. Thus, the optimal path through microfinancial state space—the asset holdings by investor—experiences discrete jumps. For that reason, we choose to regard a macrofinancial description of the system—a macrofinancial state—to be a probability distribution over all the microfinancial states; and **macrofinancial equilibrium** to be a probability distribution at a point in time that is time invariant (with respect to endogenous variables).

Within this probabilistic framework, definite answers can be established if we assume that the number of individual assets, assumed indivisible, is very large. It is this assumption which replaces that of infinitely divisible time and continuous finite trading in models that postulate a price process. In this paper, the random distribution is not postulated, but derived. Further, the random distribution refers to not a process over time, but rather a distribution at a point in time. Even in the absence of exogenous uncertainty, equilibrium is a dynamic equilibrium, characterized by a determinate distribution (effectively, risk neutral) of expected prices around a mean.

Determining the dynamic equilibrium for stocks and bonds (Section 2 and Appendix) requires some notion of what constitutes distinct microfinancial states of the system. To that end, we adopt an axiom of **indistinguishability**, meaning an inability to distinguish marginal redistributions of portfolio assets through trade. Underlying this notion is the observation that financial markets display a determinate-price equilibrium in which optimal portfolio composition is indeterminate (when prices evolve deterministically). Indistinguishability combined with allowed residency determines the enumeration of states and the assignment of a probability measure. Accordingly, one can calculate (Section 3) the equilibrium probability distributions and macrofinancial quantities, that is, probability averaged financial quantities and their distributions.

The effect of risk on probability distribution averaging (Section 4) is introduced through a set of exogenous risk parameters, fixed in number, whose value affects the value of the stocks or bonds. The parameters undergo fluctuations, assumed to be small. Additionally, investors may make random reallocations of cash between the stock and bond markets and other markets. Even though almost nothing is specified about the risk parameters, the resulting distributions are determinate and depend critically on the dimensionality of risk, that is, the number of parameters. This number is regarded as a quantity to be determined empirically.

Empirical findings (for instance, Jackwerth and Rubinstein (1996) and Kim (1999)) show that the distribution of stock prices is approximately lognormal, with systematic deviations such as left skewness and leptokurtosis; and that the distribution of corporate bond prices is approximately double exponential, interpreting short-run changes in credit quality as a realization of the distribution. *Prima facie*, the analytical expressions that we derive for the equilibrium probability distributions offer the promise of an excellent fit to the empirical data.

II. Dynamic Equilibrium for Stocks and Bonds with No Risk

Let there be given a financial system $\mathfrak{F}$, consisting of a set of assets chosen through time t by a given set of investors, denoted by index μ $(\mu = 1, \ldots, M)$, so as to optimize consumption $c(t) \equiv c \equiv (c^1, \ldots, c^M)$. $\mathfrak{F}$ is implicitly part of an economy $\mathfrak{E}$ in which firms issue and redeem liabilities that are the counterparts to the assets.

Assets are either of two kinds, stocks and bonds, denoted by respective indices S and B; we shall develop the theory in parallel for both S and B, differentiating the two only as needed. (It seems that a similar approach as for the aggregate market could be applied to an individual firm's stocks or bonds, with similar results.) Assets are of K types, denoted by index κ $(\kappa = 1, \ldots, K)$, each corresponding to a liability of a particular firm; excepting that $\kappa = 1$ denotes the money market deposit (potentially, the liability of any and all firms), which is a capital–certain asset paying a nominal rate of interest ρ. Assets $\kappa = 1, \ldots, K_B$ are bonds, and $\kappa = K_B + 1, \ldots, K$ are stocks $(K_B < K)$. d_κ denotes the current return on asset κ (the dividend or coupon rate, with $d_1 = \rho$). All assets are denominated in currency units $h > 0$, assumed to be indivisible. Asset holdings by type and investor are specified by quantity $n \equiv (n) \equiv (n^{11}, \ldots, n^{KM}) \equiv \{n^{\kappa\mu}\}$, of nominal value nh, and by corresponding marginal market price $p \equiv \{p_{\kappa\mu}\}$; $p_{1\mu} = 1$, the numeraire, and $p_{-1\mu}$ denotes the price of c^μ. $\mathfrak{U} = \{n\}$, the set of all possible (n), is called the state space; $\mathfrak{V} = \{p, n\}$, the phase space; and $\mathfrak{W} = \left\{t, n, \dot{n}\right\}$, $\dot{n} \equiv dn/dt$, we shall call the configuration space. Each consumer's unit portfolio, $(1^\mu) \equiv (1^{1\mu}, \ldots, 1^{K\mu})$, is assumed to constitute a linearly independent basis set. The formalism admits the possibility of a local market structure in $\mathfrak{F}$, across which $p_{\kappa\mu}$ may differ for fixed κ $(\neq 1)$.

Implicitly in $\mathfrak{E}$, profit maximizing firms produce and invest in such a way as to follow their optimal path of investment, and ρ is determined endogenously by the equating of aggregate savings and investment. Accordingly, firms' net cashflow on securities D—issuance less servicing of liabilities:

$$D = \sum_{\kappa} \sum_{\mu} p_{\kappa\mu}\, dn^{\kappa\mu}/dt - d_{\kappa\mu}\, n^{\kappa\mu}$$

—is determined as to size (by the aggregate budget constraint of either firms or investors), though not as to composition, since we shall maintain the assumption that firms are indifferent as to the composition of these cashflows (meaning, for instance, Modigliani-Miller indifference to debt/equity ratios). The $n^{\kappa} = \sum_{\mu} n^{\kappa\mu}$ will vary, since firms may create or redeem liabilities at will (subject to an investor creditworthiness ceiling).

Remark 1 *One could arbitrarily assign a path to either component of D and thereby fix the other: for instance, by assuming n fixed, $dn/dt = 0$ (zero net securities issuance); or the polar opposite $d_{\kappa} \equiv d_{\kappa\mu} = 0$ (say, zero dividends and no bonds outstanding). But such an assignment would necessarily affect the price process $p(t)$ in an incalculable manner, since it would imply a counterpart assignment of total return ρ between capital gains and current returns. Any such assignment is a matter of indifference in what follows, which concerns only total returns.*

Asset markets are assumed to be frictionless (apart from the indivisibility of assets) with, in particular, no transaction costs, no restrictions on short-selling, and no taxes. Investors are assumed to be price takers, and markets are assumed to clear. Investors maximize the present value of utility flows (with heterogeneous rates of time preference) through the exercise of short-run rational expectations alone. The system is nonetheless determinate, because we regard not only initial quantities $n(t_A)$, but also initial prices $p(t_A)$, as given empirical parameters.

Hence, asset prices evolve deterministically and there is no risk, in the sense that there is no short-run nominal uncertainty. The total rate of return ρ, comprising both income and capital gains, is assumed to be a function on the joint phase spaces $\{q_e, k\}$, the real economy shadow-price and capital stock, and $\{p_{-1}, c\}$, without explicit dependence on time:

$$\begin{aligned} \rho &= \rho(q_e, k; p_{-1}, c); \\ \partial\rho/\partial t &= 0, \end{aligned}$$

expressing the presumption that with a given static technology, the value of optimal investment is determined by (q_e, k) (for given p_{-1}). Total rate of return is equalized across all assets, both financial and real; thus, ρ may be regarded as a risk-neutral return in that all returns are risk free (and the issue of completeness of securities markets does not arise).

Nevertheless, these conditions only hold almost always, because dividend and coupon payments are allowed to take discrete finite values; "almost always" means at almost all points, excepting at most a set of measure zero, on the optimal time path $O^{\oplus}_{A\Omega}$ (in general, not unique) in $\mathfrak{U}$. At points of discrete cashflow, asset market equilibrium cannot be accomplished by marginal trades, thereby necessitating finite trades. As a consequence of optimal investor behaviour, therefore, trading is marginal almost always, and finite only at discrete points. (These points constitute a set of measure zero; recall from set theory, however, that their number may be denumerably infinite, or even non-denumerably infinite.)

With optimal portfolio choice, the present value of portfolio returns (the momentary rate of return) including capital gains is given by

$$Y_t''(p,n) = \rho\, h \exp\left(-\int_{t_A}^{t} \rho\, d\tau\right) \sum_{\kappa}\sum_{\mu} p_{\kappa\mu} n^{\kappa\mu}; \qquad a.e.,$$

where "a.e." denotes "almost everywhere", synonymous with "almost always". Here, the $''$ denotes a double transformation of coordinates, from current to present value prices, and from current to total return (but we write Y_t'' in terms of observable $(p_{\kappa\mu},\ n^{\kappa\mu})$).

The derivation of this expression is given in the Appendix. There, the point is made that the inclusion of capital gains so as to pass from current to total return can be justified on the basis of a time-dependent transformation of coordinates $\left(n,\ \dot{n}\right)$ in $\mathfrak{W}$; and such time-dependent coordinate transformations are necessarily allowable if one makes the assumption of the invariance (scalar nature with respect to its arguments) of the Lagrangian of the portfolio choice problem:

$$\begin{aligned} L &= L\left(t,\ n,\ \dot{n}\right) \\ &= \exp\left(-\int_{t_A}^{t} \rho\, d\tau\right) \sum_{\kappa}\sum_{\mu} \left(d_{\kappa\mu} n^{\kappa\mu} - p_{\kappa\mu} \frac{dn^{\kappa\mu}}{dt}\right), \end{aligned}$$

where investors act as if to maximize $\int_{t_A}^{t_\Omega} L\, dt$, the present value of net cashflows (that is, act as if discounting at the common observable rate ρ).

Because $\partial\rho/\partial t\ =\ 0$, Y_t'' represents a constant of the dynamical evolution of $\mathfrak{W}$ (equivalently $\mathfrak{V}$),

$$\frac{dY_t''}{dt} = 0 \quad a.e.\ on\ O_{A\Omega}^{\oplus}$$

(as shown in the Appendix), and in general the only determinate constant. Consequently, a probability density over $\mathfrak{V}$,

$$\Psi\, d\mathfrak{V} \equiv \Psi\,(p,n)\ dp_{11} \ldots dp_{KM} dn^{11} \ldots dn^{KM},$$

must take the form

$$\Psi\, d\mathfrak{V} = \Psi\,(Y_t'')\ d\mathfrak{V},$$

without explicit dependence of Ψ on (t,p,n) (equivalently, $\left(t,n,\dot{n}\right)$), if it is to represent macrofinancial equilibrium. $\Psi\,(Y_t'') \equiv \Psi\,(Y_t''\,[p,n])$ then equals the (relative) probability of occurrence of the microfinancial state (p,n) at time t. We now proceed to construct such a macrofinancial equilibrium state Ψ.

Elsewhere (Johannes, 2001), I have shown on the basis of essentially the same assumptions as will be made here, that a heuristic argument concerning the most probable distribution leads to a Ψ of the form

$$\int \Psi \, d\mathfrak{V} = a\frac{1}{\exp\left[\nu^*\right]-1} + b\int_0^\infty \frac{dP}{\exp\left[\nu^* + Y/\alpha^*\right]-1} \qquad (a, b \text{ constant})$$

for bonds and zero-return money. The first term refers to zero-return money and will not concern us further here. The second term (which we shall derive here by a different route) refers to bonds: $Y \equiv Y_t = \rho^* hP$ denotes the current value $Y = \exp\left(+\int_{t_A}^t \rho \, d\tau\right) Y''$ of an individual financial asset, and P denotes its price; a * denotes the larger systems (market, economy) of which $\mathfrak{F}$ and $\mathfrak{E}$ form a small component; α^* is the Lagrangian multiplier associated with Y^* (and is also a measure of expected nominal output in $\mathfrak{E}$); and ν^* is the Lagrangian multiplier associated with expected volume of financial assets in $\mathfrak{E}$ (equivalently, volume of financial assets in $\mathfrak{E}^*$). In this paper, using methods largely known from the mathematical and scientific literature, we shall derive the corresponding result for stocks and bonds, starting out with a brief heuristic argument and then proceeding to a rigorous treatment of the case of N assets.

Consider two composite assets 1 and 2 that are part of $\mathfrak{F}$, each consisting of all the liabilities of one kind (S or B) of respective firms 1 and 2, and with corresponding Y_1 and Y_2. The subscript t on Y is henceforth omitted, since we shall be concerned exclusively with momentary equilibrium at a point in time. Assume that these distinguishable assets form a small part of $\mathfrak{F}$, such that $Y_1, Y_2 \ll Y$ for $\mathfrak{F}$ in aggregate. In the idealized limit of many firms (identical in type though not in size) in which $Y_1/Y, Y_2/Y \to 0$, we would expect the probability densities for the two components, $\Psi\left(Y_1\right)$ and $\Psi\left(Y_2\right)$ to be statistically independent; note that this expectation asserts nothing about the assets' returns, only something about the probability of occurrence of assets bearing Y_1 and Y_2. Statistical independence implies that Ψ satisfies the functional equation

$$\Psi\left(Y_1 + Y_2\right) = \Psi\left(Y_1\right)\Psi\left(Y_2\right).$$

Let us assume that Ψ is continuous. Then we can apply the mathematical theorem that the only solutions of this equation for functions continuous on the real line are the exponential functions:

$$\Psi\left(Y\right) = c\exp\left(-Y/\alpha\right) \quad (c, \alpha \text{ independent of } Y),$$

and such functions are necessarily analytic, a fact that will be used in Section 4. Further, it is clear that first, the common discount factor $\exp\left(-\int_{t_A}^t \rho \, d\tau\right)$ in Y'' may be absorbed in the conjugate variable α, thereby justifying expressing Ψ in terms of the observable current value Y; second, we require $\alpha > 0$, so as to ensure bounded probabilities for $Y \geq 0$; third, the value of α must be the same for component systems that are in macrofinancial equilibrium; fourth, given the commonality of the value of α, we may confine our considerations in equilibrium to $Y \geq 0$, consistent with the limited liability of financial asset ownership; fifth, the constant c plays the role of the normalizing factor, and therefore may be dropped, with normalization by $1/\int_{VT} \Psi \, d\mathfrak{V}$

when required, where $\int_{VT}$ denotes an integral over the domain of $(p, n) \in \mathfrak{V}$.

The same argument applies to $\mathfrak{F}$ as a whole (that is, the markets $S + B$), if we assume that $\mathfrak{F}$ is a small component of a larger set of asset markets, not only the financial assets in $\mathfrak{F}^*$, but also the real asset markets in $\mathfrak{E}$ and $\mathfrak{E}^*$. We conclude, therefore, that the distribution of financial assets in $\mathfrak{F}$ may be described by the normalized probability density

$$\Psi(Y)\, d\mathfrak{V} = \frac{\exp\left(-Y/\alpha^*\right)\, d\mathfrak{V}}{\int_{VT} \exp\left(-Y/\alpha^*\right)\, d\mathfrak{V}} \quad (\alpha^* > 0),$$

where α^* is determined outside of $\mathfrak{F}$ (by equilibrium in $\mathfrak{F}^*$ and $\mathfrak{E}^*$). In the event that Y takes not continuous, but discrete values $Y_\lambda, \lambda = 1, 2, \ldots$, the probability law assumes the form

$$\Psi(Y_\lambda) = \frac{\exp\left(-Y_\lambda/\alpha^*\right)}{\sum_\lambda \exp\left(-Y_\lambda/\alpha^*\right)} \quad (\alpha^* > 0;\ \lambda = 1, 2, \ldots).$$

On this basis, let us turn to the problem of determining Ψ for N individual assets (distinguishing N^S and N^B as needed). Hidden in the foregoing argument has been the presumption that the composite assets are distinguishable, that is, that each asset may be specified precisely as to its constituent types $\kappa = k_1, k_2, \ldots$ and quantities $n^{k_1}, n^{k_2}, \cdots$. For individual assets, that presumption may be doubted. For, optimal investor behaviour in $\mathfrak{F}$ precisely determines the path of prices $p(t)$, but leaves indeterminate the path of portfolio composition $n(t)$ (as shown in the Appendix). Such behaviour accords with the notion that financial assets are subject to a no arbitrage principle, whereby deviations from precise equilibrium prices are eliminated through large portfolio purchases and sales of indeterminate quantity.

It seems reasonable to suppose, therefore, that at equilibrium prices, one cannot distinguish marginal differences arising from trade between investor portfolios. To do so would require information on market transactions that could not be obtained without disturbing the equilibrium prices concerned (with the presumption that investors eschew voluntary disclosure of their transactions). We shall embody this supposition in the form of the following axiom of indistinguishability.

Axiom 1 *Consider a financial system $\mathfrak{F}$, consisting of a given set of utility maximizing asset traders $\{\mu\}$ and financial assets of various types $\{\kappa\}$, and denote marginal market asset prices by $p_{\kappa\mu}$. Suppose further that at a given time traders establish determinate values (i.e. precisely specified values) for the $p_{\kappa\mu}$ through market transactions. Then, we regard as indistinguishable states of $\mathfrak{F}$ at that time that differ only by a marginal redistribution of assets through trade.*

The economic interpretation is that for the aggregate portfolio n in $\mathfrak{F}$, marginally differing states n_1 and n_2 of equal trader utility, as evidenced by agreed marginal prices $p_{\kappa\mu}$, are regarded as indistinguishable. The application to our $\mathfrak{F}$ is that the $p_{\kappa\mu}$ evolve in a determinate way, and so a portfolio n is regarded as one and the same state regardless of the marginal redistribution of individual assets (that is, assets of nominal quantity h). Redistribution refers to not only ownership μ, but also type κ, since firms may issue and retire liabilities. In other words, the axiom implies that we cannot attach a label (κ, μ) to an individual asset.

Accordingly, individual assets are identified simply by price P_ι, and the aggregate portfolio n is completely specified by the set of numbers of individual assets N^ι at each price, of total number $N = \sum N^\iota$. We denote a specific set by $(N^1_\lambda, N^2_\lambda, \cdots)$. We shall start by considering discrete single–asset price levels $P_\iota, \iota = 1, 2, \ldots$, assumed fixed, and then pass to the quasi-continuous case $P_\iota \to P$ (at which point, the density of states or state price density dV/dP will be introduced).

Consider a market (S or B) of N individual assets with observed current return $\widehat{Y}$, where a $\widehat{\ }$ denotes an observed quantity (empirical datum). By virtue of the axiom of indistinguishability, a single state of the market is specified by $(N^1, N^2, \cdots)$ with corresponding $Y = \rho^* h \sum_\iota P_\iota N^\iota$, where ρ^* is determined in $\mathfrak{E}^*$. The potential values Y_λ of Y, each of which is defined by a set of values $(N^1_\lambda, N^2_\lambda, \cdots)$,

$$Y_\lambda = \rho^* h \sum_\iota P_\iota N^\iota_\lambda$$

for given ρ^* and $\{P_\iota\}$, will follow the distribution law

$$\Psi\left(Y_\lambda\right) = \frac{\exp\left(-Y_\lambda/\alpha^*\right)}{\sum_\lambda \exp\left(-Y_\lambda/\alpha^*\right)} = \frac{\exp\left(-\left[\rho^* h/\alpha^*\right] \sum_\iota P_\iota N^\iota_\lambda\right)}{\sum_\lambda \exp\left(-\left[\rho^* h/\alpha^*\right] \sum_\iota P_\iota N^\iota_\lambda\right)}.$$

To comprehend the empirical significance of Ψ, we must express it in terms of accessible quantities through direct calculation.

III. Calculation of the Probability Distribution Ψ and Macrofinancial (Ψ–Averaged) Quantities

For the calculation of Ψ and associated financial quantities, it is convenient to define the moment generating function by

$$\Phi\left(\alpha^*\right) = \sum_\lambda \exp\left(-\frac{\rho^* h}{\alpha^*} \sum_\iota P_\iota N^\iota_\lambda\right), \tag{1}$$

where the P_ι are regarded as given implicit parameters, $\Phi\left(\alpha^*\right) \equiv \Phi\left(\alpha^*; \{\rho^* h P_\iota\}\right)$; the successive logarithmic derivatives of Φ with respect to α^* generate the corresponding central moments of Ψ. From $\log \Phi$ can be derived observable quantities of interest pertaining to the macrofinancial state, that is, probability averaged (or "phase averaged") quantities: for example,

$$\langle N^\iota \rangle = -\frac{\alpha^*}{\rho^* h} \frac{\partial}{\partial P_\iota} \log \Phi, \tag{2}$$

the average number of assets bearing price P_ι, where $\langle\,\rangle$ denotes a probability average (the mathematical expectation) and log the natural logarithm; and

$$\langle Y \rangle = \left(\alpha^*\right)^2 \frac{\partial}{\partial \alpha^*} \log \Phi, \tag{3}$$

the expected value of Y in $\mathfrak{F}$, which incidentally shows that α^* is determined by the value of $Y = Y^*$ in $\mathfrak{F}^*$ (and in fact can be related to nominal output in $\mathfrak{E}^*$ on the basis of the commonality of α^* in equilibrium noted above). With the postulate that observed macrofinancial quantities are identified with probability averages, we have that $\langle Y \rangle = \widehat{Y}$, and hence $\widehat{\alpha^*}$ can in principle be inferred from $\widehat{Y}$ measured in $\mathfrak{F}$ (given a specific model for $\mathfrak{F}$).

In the definition of $\Phi(\alpha^*)$, the $\sum_\lambda$ signifies the sum over all admissible sets of $(N_\lambda^1, N_\lambda^2, \cdots)$. Let us assume that the total number of individual assets N for the market (S or B) is fixed. The underlying rationale is that the sum function $\langle N \rangle = \sum \langle N^\iota \rangle$ is time invariant (by construction), and for large N

$$N \to \infty,$$

fluctuations in N (the distribution around the mean) will become negligible assuming, for example, that the variance of N increases no faster than N itself. Hence $N \simeq \langle N \rangle$, a constant. Then the admissible sets $(N_\lambda^1, N_\lambda^2, \cdots)$ are those that satisfy

$$\sum_\iota N_\lambda^\iota = N, \tag{4}$$

and permutations of individual assets within a given N_λ^ι are regarded not as giving rise to different states but as part of the same N–asset state λ.

The system $\mathfrak{F}$ will be completely specified macrofinancially once we stipulate one final quantity: the allowed residency numbers for each individual-asset state ι. Reflecting the considerations of section 1, we adopt the following axiom.

Axiom 2 *Consider a financial system $\mathfrak{F}$ comprising markets S and B, with respectively $N = N^S$ and $N = N^B$ individual financial assets of denomination h, respectively individual-asset states $\{\iota^S\}$ and $\{\iota^B\}$, and respectively support by underlying real assets and underlying fiat assets. For individual assets in S, the allowed residency $N^\iota = N^{S\iota}$ in a given state ι^S is at most one ($N^{S\iota} = 0, 1$); for individual assets in B, the allowed residency $N^\iota = N^{B\iota}$ in a given state ι^B is unlimited ($N^{B\iota} = 0, 1, 2, \ldots$).*

Strictly speaking, the allowed residency applies to the underlying assets and their states, since financial assets represent claims on a cross–section of all underlying assets; for our present accounting purposes, however, we may regard the financial assets as being built up by marginal additions, with corresponding additions to the underlying assets. We shall ignore the possibility of hybrid real and fiat assets, for example convertible bonds, which may be regarded as assets whose expected quantity cannot be time-invariant under dynamic optimization.

The problem of finding the distributions of $\langle N^\iota \rangle$ with P_ι for S and B, that is, the equilibrium distributions of value, is thus reduced to calculation of $\Phi(\alpha^*)$ in accordance with its defining equation (1), subject to the condition (4) and the two-part condition,

$$
\begin{aligned}
N_\lambda^\iota &= \begin{cases} 0,1 & (S) \\ 0,1,2,\ldots & (B)\,, \end{cases} \\
\iota &= 1,2,\ldots\,;\, \lambda = 1,2,\ldots\,,
\end{aligned}
\tag{5}
$$

where the notation (S) and (B), it is to be hoped, is self-explanatory. In essence, the axiom on allowed residency by financial assets supported by either real or fiat assets amounts to stipulating that they follow respectively the so-called Fermi-Dirac or Bose-Einstein statistics, familiar in the scientific literature: Feller (1968) ch. II, which cites the example that *a priori* one might expect typographical errors on a printed page to follow Fermi-Dirac statistics.

As an illustration of the difference between the different ways of reckoning Ψ, consider the following example. Suppose that there is given a single asset, of denomination h and a certain type, that may assume either of only two states, P_{HI} and P_{LO}; and that there is an equal probability $1/2$ of either outcome. Further, suppose that there is given a second single asset of the same type. When assets are distinguishable, there is a probability $1/4$ of both assets residing in the same state, in accordance with the rule for independent events of multiplying probabilities. In S, the probability of both assets residing in the same state is 0; in B, it is $1/3$, since there are only three independent states $(P_{HI}P_{HI},\ P_{HI}P_{LO} \equiv P_{LO}P_{HI},\ P_{LO}P_{LO})$ when assets are indistinguishable. This example verifies the assertion made earlier that the assumption of distinguishability is implicit in the argument of statistical independence.

Writing

$$
\begin{aligned}
\Phi\left(\alpha^*\right) &= \sum_\lambda z_1^{N_\lambda^1} z_2^{N_\lambda^2} \cdots z_\iota^{N_\lambda^\iota} \cdots\,; \\
\text{where} \quad z_\iota &= \exp\left(-\frac{\rho^* h}{\alpha^*} P_\iota\right) \leq 1; \qquad \iota = 1,2,\ldots\,;\, \lambda = 1,2,\ldots\,,
\end{aligned}
$$

we see that (4) will be satisfied provided that Φ consists solely of all terms that are homogeneous of degree N $(= N^S$ or $N^B)$ in the $\{z_\iota\}$. Because the N_λ^ι appear as exponents in Φ, (5) generates as potential combinations in $\sum_\lambda$ a product of geometric progressions (of which, the homogeneous terms are to be retained). In the ιth factor of potential combinations, there appears for S, $(1 + z_\iota)$, and for B, $(1 + z_\iota + z_\iota^2 + \cdots) = 1/(1 - z_\iota)$. Hence, for Φ we obtain the expressions

$$
\begin{aligned}
&\prod_\iota (1 + z_\iota) && (S) \\
&\prod_\iota \frac{1}{(1 - z_\iota)} && (B)\,,
\end{aligned}
$$

subject to the requirement that out of the product $\prod_\iota$ one retains solely terms that are homogeneous of degree N in the $\{z_\iota\}$.

Stated thus, we see that we can apply Cauchy's integral formula to obtain explicit expressions for Φ. Working in the complex plane, let us introduce the auxiliary multiplier β^*

through the two-part definition

$$w(\beta^*) = \prod_\iota (1 + \beta^* z_\iota) \qquad (S)$$
$$w(\beta^*) = \prod_\iota \frac{1}{(1 - \beta^* z_\iota)} \qquad (B),$$

where β^* is a complex variable that plays a role similar to that of a Lagrange multiplier for the condition (4), with the * notation anticipating that β^* is determined in $\mathfrak{F}^*$. Then Φ is given exactly by an integral around a closed contour in the complex β^* plane that encircles the origin $\beta^* = 0$ (but no other singularity of the integrand, and is assumed to lie within the circle of convergence of $w(\beta^*)$):

$$\Phi = \frac{1}{2\pi i} \oint \frac{w(\beta^*)}{\beta^{*N+1}} \, d\beta^* \tag{6}$$

$(i \equiv +\sqrt{-1}; \beta^{*N+1} \equiv (\beta^*)^{N+1})$.

The evaluation of the contour integral (6), a formidable task in general, is greatly helped by two key simplifications: that $N \to \infty$ (by assumption), which permits the use of asymptotic methods; and that the integrand has a unique minimum (as we shall show), which permits evaluation at that point alone. Consequently, the integral can be worked out by the saddle point method (Bleistein and Handelsman, 1975, ch. 7, sect. 2), which in its more general form of the method of steepest descents was originated by B. Riemann and P. Debye.

Consider the behaviour of the integrand in (6), $w(\beta^*)/\beta^{*N+1}$, for β^* real and positive. For $\beta^* \to 0$, the integrand is indefinitely large and positive, for both S and B. The logarithmic first derivatives are given by

$$\frac{d}{d\beta^*} \log \frac{w(\beta^*)}{\beta^{*N+1}} = -\frac{N+1}{\beta^*} + \sum_\iota \frac{z_\iota}{(1 + \beta^* z_\iota)} \qquad (S)$$
$$\frac{d}{d\beta^*} \log \frac{w(\beta^*)}{\beta^{*N+1}} = -\frac{N+1}{\beta^*} + \sum_\iota \frac{z_\iota}{(1 - \beta^* z_\iota)} \qquad (B),$$

which, for $\beta^* \to 0$, are both large and negative. As β^* increases, they both increase. For B, assuming for simplicity that the lowest price level is $P_\iota = 0$, as $\beta^* \to 1$, the derivative becomes indefinitely large and positive (for fixed large N). For S, as $\beta^* \to +\infty$, the derivative becomes positive and remains so. To see this, note that its limiting value becomes $(1/\beta^*)\{-(N+1) + \sum_\iota 1\}$, and we have $\sum_\iota 1 > (N+1)$, since $\sum_\iota 1$ represents the number of states, some of which will be empty by virtue of (5).

Therefore (since log constitutes a monotonic transformation), $w(\beta^*)/\beta^{*N+1}$ possesses a unique minimum, for both S and B. We denote the corresponding (real) value of β^* by $\beta^{*\oplus}$, which is determined by setting $d\left(w(\beta^*)/\beta^{*N+1}\right)/d\beta^* = 0$ (the first order conditions, which we shall return to shortly). The values of the second derivatives, which we shall denote by σ^2, are given by

$$\sigma^2 = \frac{d^2}{d\beta^{*2}} \log \frac{w(\beta^*)}{\beta^{*N+1}} = \frac{N+1}{\beta^{*2}} + \sum_\iota \frac{-z_\iota^2}{(1+\beta^* z_\iota)^2} \qquad (S)$$

$$\sigma^2 = \frac{d^2}{d\beta^{*2}} \log \frac{w(\beta^*)}{\beta^{*N+1}} = \frac{N+1}{\beta^{*2}} + \sum_\iota \frac{z_\iota^2}{(1-\beta^* z_\iota)^2} \qquad (B),$$

and, unless qualified, we shall mean the stationary point value $\sigma^2 = \sigma\left(\beta^{*\oplus}\right)^2 > 0$. (Case S, for which we may have $\beta^* \to +\infty$ and $\sigma^2 < 0$, illustrates the point that, in order to establish the existence of a unique minimum or maximum, it may be neither necessary—case S—nor sufficient that the second derivative be always positive or negative, respectively; a trivial example of insufficiency being $\log x,\ 0 < x < +\infty$, which has no maximum.)

Assuming that σ^2 is very large of the order of N ($\sigma^2 \sim N$, which we shall verify below), the saddlepoint method shows that the value of (6) consists in the contribution from that point alone, with the dominant term derived from a Taylor series expansion of $\log w(\beta^*)/\beta^{*N+1}$ in the neighbourhood of $\beta^{*\oplus}$,

$$\log w(\beta^*)/\beta^{*N+1} = \log w\left(\beta^{*\oplus}\right)/\beta^{*\oplus N+1} + \frac{1}{2!}\left(\beta^* - \beta^{*\oplus}\right)^2 \sigma^2 + \cdots,$$

to give:

$$\Phi(\alpha^*) = \frac{1}{2\pi i}\frac{w\left(\beta^{*\oplus}\right)}{\beta^{*\oplus N+1}} \int_{-\infty}^{+\infty} \exp\left(-\frac{1}{2}\eta^2\sigma^2\right) i\, d\eta$$

where η represents the imaginary part of β^*. (In intuitive terms, at the unique saddle point $\beta^{*\oplus} = \beta^{*\oplus} + i0$, the modulus of the integrand attains a maximum, and its phase is stationary for a small, purely imaginary change $i\,d\eta$, that is, a contour that cuts the real axis orthogonally at that point. In the limit $N \to \infty$, the contribution from the neighbourhood of $\beta^{*\oplus}$ dominates all others, which are self-cancelling because of infinitely rapid oscillations of phase.) Hence, $\Phi(\alpha^*) = \left(w\left(\beta^{*\oplus}\right)/\beta^{*\oplus N+1}\right)\left(1/\sqrt{2\pi\sigma^2}\right)$ and

$$\log\Phi(\alpha^*) = -(N+1)\log\beta^{*\oplus} + \log w\left(\beta^{*\oplus}\right) - \frac{1}{2}\log\left(2\pi\sigma^2\right).$$

Under the maintained assumption of $N \to \infty$, we can set $(N+1) \simeq N$, and neglect the term $\log(2\pi\sigma^2)$, since $(1/N)\log N \to 0$ as $N \to \infty$. Evidently, N very large requires that $N \gg \log N$. It is clear that in many circumstances, N being large cannot be regarded as a self-contained fact, but rather is a consequence of M, the number of investors, being large. We arrive at

$$\log \Phi\left(\alpha^*, N\right) = \begin{cases} -N \log \beta^{*\oplus} + \sum_\iota \log\left(1 + \beta^{*\oplus} \exp - \left[\rho^* h/\alpha^*\right] P_\iota\right) & (S) \\ -N \log \beta^{*\oplus} - \sum_\iota \log\left(1 - \beta^{*\oplus} \exp - \left[\rho^* h/\alpha^*\right] P_\iota\right) & (B), \end{cases}$$

reintroducing the observable prices P_ι (with single-asset $Y_\iota = \rho^* h P_\iota$). $\Phi\left(\alpha^*, N\right)$ becomes a function of the parameter N by virtue of the counterpart constraining condition, and $\beta^{*\oplus}$ enters the expression for Φ.

From the first order conditions noted above, $d\left(w\left(\beta^*\right)/\beta^{*N+1}\right)/d\beta^* = 0$, and setting $(N+1) \simeq N$, we infer that

$$\langle N \rangle = \begin{cases} \sum_\iota 1/\left\{\exp\left(\nu^* + \left[\rho^* h/\alpha^*\right] P_\iota\right) + 1\right\} & (S) \\ \sum_\iota 1/\left\{\exp\left(\nu^* + \left[\rho^* h/\alpha^*\right] P_\iota\right) - 1\right\} & (B) \end{cases} \tag{7}$$

where we write $\beta^{*\oplus} = \exp\left[-\nu^*\right]$ for notational convenience, and we set $N = \langle N \rangle$, which was tacitly the case hitherto (recall the remarks leading to (4)), $\Phi = \Phi\left(\alpha^*, \langle N \rangle\right)$. These equations may be thought of as determining $\langle N \rangle$ given ν^* (from $\mathfrak{F}^*$), or alternatively, as determining $\widehat{\nu^*}$ given observed $\widehat{N}$ in $\mathfrak{F}$, with $\langle N \rangle = \widehat{N}$ (and α^* and the $\rho^* h P_\iota$ considered as given parameters). Additionally, applying (2) to the expressions for $\log \Phi$, we obtain

$$\langle N^\iota \rangle = \frac{1}{\exp\left(\nu^* + \left[\rho^* h/\alpha^*\right] P_\iota\right) + 1} \qquad (S)$$

$$\langle N^\iota \rangle = \frac{1}{\exp\left(\nu^* + \left[\rho^* h/\alpha^*\right] P_\iota\right) - 1} \qquad (B)$$

from which it is clear that (7) represents the discrete probability distributions Ψ (normalized to $\langle N \rangle$) over states defined by price P_ι, $\langle N \rangle = \sum_\iota \langle N^\iota \rangle$.

Let us pass to the quasi-continuous limit

$$\sum_\iota \langle N^\iota \rangle \rightarrow \int_0^{\overline{N}} \int_0^{\overline{Y}/\rho^* h} \langle N(P) \rangle \frac{dV}{dP} dP.$$

Here $N(P)$ denotes the residency of state P (the analogue of N^ι). $\overline{N} \geq N$ denotes the upper bound of N, whose existence is ensured by the assumption that n is bounded by $\underline{n}$, $\overline{n}$ (Appendix), and $\overline{N}$ may be interpreted as a counterparty creditworthiness constraint on investors. $P \leq \overline{Y}/\rho^* h$ has an upper bound, given $h > 0$ indivisible, because when wealth is bounded, which we assume, then $Y \leq \overline{Y}$ is bounded almost always.

$dV(P)/dP$ denotes the state price density; it is this quantity that constitutes the analogue to given $\{P_\iota\}$, since what matters in the quasi-continuous case is (relatively) how many states occur in a small price interval dP. $dV(P)/dP$ is defined by way of the definition of the density of states $dV^\circ(Y)/dY$ as follows. $V^\circ(Y)$ denotes the volume of single-asset phase space for portfolio returns y up to Y, $y\left(\widetilde{N},\widetilde{P}\right)=y\left(\widetilde{P}\right)<Y$, where the dummy integration variable $\widetilde{P}$ specifies price and $\widetilde{N}$ specifies the arbitrarily assigned ordering of a single asset in a portfolio ($\widetilde{N}\leq\overline{N}$ is assumed to vary over a range sufficiently large as to be quasi-continuous). $V^{\circ}(Y)$ (normalized to a total number of states $\overline{N}$) is defined by

$$V^{\circ}(Y)=\overline{N}\frac{\int_{y<Y}d\mathfrak{V}}{\int_{y<\overline{Y}}d\mathfrak{V}}=\overline{N}\frac{\int_0^{\overline{N}}d\widetilde{N}\int_0^{Y/\rho^*h}d\widetilde{P}}{\int_0^{\overline{N}}d\widetilde{N}\int_0^{\overline{Y}/\rho^*h}d\widetilde{P}}=\overline{N}\frac{Y}{\overline{Y}},$$

where the linearity of V° in Y is a consequence of the positively linear homogeneity (actually, for financial assets, linearity) of y in price $\widetilde{P}$:

$$y\left(\widetilde{N},\widetilde{P}\right)<Y\Rightarrow\frac{1}{\rho^*h}y\left(\widetilde{N},\frac{\widetilde{P}}{P}\right)<1.$$

It is the degree of homogeneity interacting with the dimensionality of risk that will prove decisive in determining the effect of exogenous risk on the probability distribution (Section 4).

The equivalent state price density is defined by the circumstance $V(P)=V^{\circ}(Y)$, implying that

$$\frac{dV(P)}{dP}=\frac{dV^{\circ}(Y)}{dY}\frac{dY}{dP}=\frac{\rho^*h}{\overline{Y}}\overline{N},$$

that is, dV/dP is constant for financial assets. The transition from discrete to continuous rests on the notion that a single-asset state is specified by an infinitesimal element $d\mathfrak{V}\equiv d\widetilde{N}\,d\widetilde{P}$ of the single-asset phase space, and consequently the number of states in a given price interval is measured by (strictly speaking, proportional to) the corresponding volume. (A deeper justification is provided in Johannes (2001), namely that the volume of phase space represents the unique Haar-von Neumann invariant measure over the dynamical evolution of $\mathfrak{F}$.)

We arrive at the quasi-continuous form of (7):

$$\langle N\rangle=\begin{cases}\left(\rho^*h\overline{N}/\overline{Y}\right)\int_0^{\overline{Y}/\rho^*h}1/\left\{\exp\left(\nu^*+\left[\rho^*h/\alpha^*\right]P\right)+1\right\}dP & (S)\\ \left(\rho^*h\overline{N}/\overline{Y}\right)\int_0^{\overline{Y}/\rho^*h}1/\left\{\exp\left(\nu^*+\left[\rho^*h/\alpha^*\right]P\right)-1\right\}dP & (B).\end{cases}\tag{8}$$

We see that $\langle N\rangle$ is directly proportional to $\overline{N}$, in accordance with the natural suppositions that $\langle N\rangle$ will vary in proportion to the size of $\mathfrak{F}$ in some sense, and that $\overline{N}$ is a measure of the size of $\mathfrak{F}$. Therefore, we can now assign a more precise meaning to the limit $\langle N\rangle\to\infty$, namely, we stipulate that as $\langle N\rangle\to\infty$, the ratio $\langle N\rangle/\overline{N}$ is subject to a fixed upper bound. Consequently, ν^* remains bounded below and $\beta^{*\oplus}$ above, as $\langle N\rangle\to\infty$. We thus verify (from the earlier derived expression for σ^2) that σ^2 is very large of the order of N as $\langle N\rangle\to\infty$.

We may identify $\langle N\rangle$ with $\int\Psi\,dV=\int\Psi\,(dV/dP)\,dP$ (normalized to $\langle N\rangle$); that is, Ψ

represents the (relative) probability weight (expected residency) of a given state:

$$\Psi(Y) \equiv \Psi(\rho^* hP) = \begin{cases} 1/\{\exp(\nu^* + [\rho^* h/\alpha^*]\, P) + 1\} & (S) \\ 1/\{\exp(\nu^* + [\rho^* h/\alpha^*]\, P) - 1\} & (B). \end{cases} \tag{9}$$

The equilibrium distribution of value (equivalently, price) with no exogenous risk is given by the integrand in

$$\begin{aligned} \langle Y \rangle &= \frac{\rho^* h}{\overline{Y}} \overline{N} \int_0^{\overline{Y}/\rho^* h} \frac{\rho^* h\, P}{\exp(\nu^* + [\rho^* h/\alpha^*]\, P) + 1}\, dP \qquad (S) \\ \langle Y \rangle &= \frac{\rho^* h}{\overline{Y}} \overline{N} \int_0^{\overline{Y}/\rho^* h} \frac{\rho^* h\, P}{\exp(\nu^* + [\rho^* h/\alpha^*]\, P) - 1}\, dP \qquad (B). \end{aligned}$$

For (S), $\nu_S^* \in (-\infty, +\infty)$, and for (B), $\nu_B^* \in (0, +\infty)$. For both (S) and (B), the distribution of value (or price) tends to zero as $P \to 0$ (even if $\nu^* = 0$), and so the integrals converge. (As a technical matter, the values of these integrals for $\nu^* = 0$ and with the upper bound infinite can be expressed in terms of the gamma and Riemann zeta functions through integration by parts.)

Finally, we see again that the informational content of Ψ, $\langle N \rangle$, and $\langle Y \rangle$ can be deduced from knowledge of

$$\log \Phi(\alpha^*, \langle N \rangle) \tag{10}$$

$$= \begin{cases} \langle N \rangle\, \nu^* + \left(\rho^* h/\overline{Y}\right) \overline{N} \int_0^{\overline{Y}/\rho^* h} \log\left(1 + \exp - (\nu^* + [\rho^* h/\alpha^*]\, P)\right) dP & (S) \\ \langle N \rangle\, \nu^* - \left(\rho^* h/\overline{Y}\right) \overline{N} \int_0^{\overline{Y}/\rho^* h} \log\left(1 - \exp - (\nu^* + [\rho^* h/\alpha^*]\, P)\right) dP & (B), \end{cases}$$

where quasi-continuous $\log \Phi$ is implicitly dependent on the quasi-continuous price structure $dV(P)/dP$. For instance, $\int \Psi\, dV$ takes the form

$$\int \Psi\, dV = \int \Psi \frac{dV}{dP}\, dP \equiv -\frac{\partial \mathfrak{L}\left(\log \Phi\left(\langle N \rangle; \cdot\right)\right)}{\partial \nu^*},$$

where $\mathfrak{L}$ denotes the Legendre transform:

$$\mathfrak{L}\left(\log \Phi\left(\langle N \rangle; \cdot\right)\right) = \log \Phi - \langle N \rangle\, \partial \log \Phi / \partial \langle N \rangle.$$

Thus, the macrofinancial state of $\mathfrak{F}$ is specified by the macrofinancial parameters α^* and $\langle N \rangle$ (or ν^*, equivalent to $\langle N \rangle$, by virtue of (8)).

IV. Exogenous Risk and its Effect on Probability Distribution (Ψ) Averaging

The price structure in $\mathfrak{F}$, $\{P_\iota\}$ or its quasi-continuous counterpart $dV(P)/dP$, we have regarded thus far to be fixed. Consider now circumstances in which prices P, and consequently price structure dV/dP, will vary with fluctuations $(\delta) \equiv \delta\chi$ in exogenous observable parameters

$\chi = \left(\chi^1, \dots, \chi^L\right)$ in $\mathfrak{E}$, and with shifts in cash allocations to $\mathfrak{F}$ from $\mathfrak{F}^*$, giving rise to shifts $(\Delta) \equiv \Delta\alpha^*$, $\Delta \langle N \rangle$ (equivalently, $\Delta\nu^*$) in the macrofinancial parameters. The (δ), but not the (Δ), are assumed to be small (in the sense of justifying a Taylor expansion to first order only). More generally, we may speak of (δ, Δ) in $\mathfrak{E}^*$.

The exogenous parameters and cash allocations are assumed to change not continuously, but at discrete points in time (discrete cash trades being consistent with the considerations of Section 2 and the Appendix applied to $\mathfrak{F}^*$). Accordingly, following the occurrence of a given set of (δ, Δ), we can envisage a period of time sufficiently long that $\mathfrak{F}$ comes into internal equilibrium through market trading, yet sufficiently short that the given (δ, Δ) continue to hold, and the state of $\mathfrak{F}$ constitutes a temporary deviation from its previous equilibrium with $\mathfrak{E}^*$. In mathematical terms, interactions of $\mathfrak{F}$ with external markets $\mathfrak{E}^*$ constitute a set of measure zero (through time), and hence external conditions may be considered fixed almost always. In this way, we can give a rational meaning to the notion of equilibrium in the subsystem $\mathfrak{F}$ following occurrence of a set of (δ, Δ).

The post–(δ, Δ) ("risky") equilibrium is therefore completely described by $\int \Psi \, dV$ (equivalently, $\log \Phi$), provided that we admit given $\Delta\alpha^*$, $\Delta \langle N \rangle$ (or $\Delta\nu^*$), and $\delta\chi$. Our method of characterizing risky equilibrium consists in calculating Ψ under the assumption that the (δ) are small, and from its functional form inferring the changes in P and dV/dP, and thus the risky distribution of P. We shall first develop the formalism and then give the financial interpretation.

Let there be given exogenous risk parameters $\chi = \left(\chi^1, \dots, \chi^L\right)$, χ real, pertaining to a particular market S or B, where fixed integer L denotes the dimensionality of risk, in general, different for each market, $L_S \neq L_B$. (Recall that bonds are of indefinite maturity, and therefore we interpret χ for B as credit risk; we have not differentiated the treatment of assets in B by maturity because we wish to bypass issues of term structure.) The variation of P with χ^λ for asset κ

$$\left(\frac{\partial P}{\partial \chi^\lambda}\right)_\kappa \qquad (\lambda = 1, \dots, L, \; \kappa = 1, \dots, K)$$

is assumed to be a determinate quantity, where $(\,)_\kappa$ denotes that the quantity enclosed in parentheses pertains to assets of type κ (and for the numeraire $\left(\partial P/\partial \chi^\lambda\right)_1 = 0$). By "determinate" we mean that there exists a function determining the quantity, but we do not assume that the function is known.

We continue to presume that the type κ cannot be identified for any individual asset, and accordingly the risky single–asset distribution $\int \Psi \, dV$ that we seek must be expressed independent of any particular κ. Now the single–asset $\partial P/\partial \chi^\lambda$ is a weighted average of the $\left(\partial P/\partial \chi^\lambda\right)_\kappa$, weighted by the corresponding κ–asset distributions Ψ_κ. In turn, the Ψ_κ are determined by the normalization $\int \Psi_\kappa \, dV_\kappa = \langle N^\kappa \rangle$, where the amounts $\langle N^\kappa \rangle$ of assets κ are assumed to be observed data. Hence, single–asset $\partial P/\partial \chi^\lambda$ and the corresponding $\partial Y/\partial \chi^\lambda$ are determinate. It is convenient, therefore, to define the determinate

$$\eta_\lambda = \left(\frac{\partial Y}{\partial \chi^\lambda}\right) \qquad (\lambda = 1, \dots, L).$$

Further, we assume that in principle the $\left(\widehat{\partial P/\partial \chi^\lambda}\right)_\kappa$ are observable quantities (again, without presumption of identifying an individual asset).

In general, we would expect ρ^* to vary with the χ^λ; that variation, however, may be subsumed in a common $\Delta\rho^* = \sum_\lambda \left(\partial\rho^*/\partial\chi^\lambda\right) \Delta\chi^\lambda$, and $(\rho^* + \Delta\rho^*)$ is regarded as a given constant by virtue of our assumption of equilibrium in $\mathfrak{F}$. We have in mind the case of uniform $\Delta\rho^*$ and $\Delta\alpha^*$ for stocks and bonds, constituting equilibrium throughout $\mathfrak{F}$, though the formalism can equally be adapted to the case $\Delta\rho_S^* \neq \Delta\rho_B^*$, and similarly for $\Delta\alpha^*$.

Because of the given fluctations (δ, Δ)—treated as fixed parameters—we need to re–cast single–asset $\int \Psi \, dV$ in terms of the risky phase space for $\mathfrak{F}$, which will extend over the random variables, single-asset N and P. Since the only direct effect on (N, P) is through η_λ, we can express the marginal volume of risky phase space, dV^δ, as

$$dV^\delta = dN \, dP \; d\eta_1 \cdots d\eta_L.$$

Accordingly, from (9), the (integrated) risky distribution $\int \Psi^{(\delta,\Delta)} \, dV^\delta$ takes the form

$$\int \Psi^{(\delta,\Delta)} \, dV^\delta =$$

$$\begin{cases} \int 1/\left\{\exp\left[\nu^* + \Delta\nu^* + Y^{(\delta,\Delta)}/\left(\alpha^* + \Delta\alpha^*\right)\right] + 1\right\} dV^\delta & (S) \\ \int 1/\left\{\exp\left[\nu^* + \Delta\nu^* + Y^{(\delta,\Delta)}/\left(\alpha^* + \Delta\alpha^*\right)\right] - 1\right\} dV^\delta & (B) \end{cases}$$

where

$$Y^{(\delta,\Delta)} = (\rho^* + \Delta\rho^*) \, hP + \sum_\lambda \eta_\lambda \, \delta\chi^\lambda.$$

It is clear that $Y^{(\delta,\Delta)}$ is linear homogeneous (positively) with

$$(P, \eta) = (P, \eta_1, \cdots, \eta_L):$$

$$Y^{(\delta,\Delta)}(a\,P, a\,\eta) = a\,Y^{(\delta,\Delta)}(P, \eta); \quad a > 0.$$

Observe that $\langle P \rangle$ increases inversely with $\Delta\rho^*$, with other macrofinancial parameters constant, because as $\Delta\rho^*$ decreases, the weight of the distribution is shifted to higher P. Equivalently, we may observe that the distribution of $Y^{(\delta,\Delta)}$ is independent of $(\rho^* + \Delta\rho^*)$, and hence if the latter changes, then P changes in an offsetting fashion.

We see that

$$\Psi^{(\delta,\Delta)}\left(Y^{(\delta,\Delta)};\alpha^{*}+\Delta\alpha^{*};\nu^{*}+\Delta\nu^{*}\right)$$

takes an identical form to $\Psi\left(Y;\alpha^{*};\nu^{*}\right)$. Further, we see that, since the $\delta\chi^{\lambda}$ are small, the range of $Y^{(\delta,\Delta)}$ will take the same functional form $\left(0,\overline{Y^{(\delta,\Delta)}}/\left(\rho^{*}+\Delta\rho^{*}\right)h\right)$ as does Y, provided we assume that for $Y^{(\delta,\Delta)}\rightarrow 0$, $\sum_{\lambda}\eta_{\lambda}\,\delta\chi^{\lambda}\geq 0$, consistent with limited liability. Therefore,

$$\int\Psi^{(\delta,\Delta)}\,dV^{\delta}=\int\Psi^{(\delta,\Delta)}\,\frac{dV^{\delta}}{dY^{(\delta,\Delta)}}\,dY^{(\delta,\Delta)}$$

takes the same functional form as does $\int\Psi\,dV$ (with allowance for the terms $\Delta\alpha^{*}$, $\Delta\nu^{*}$ and $\Delta\rho^{*}$), excepting to the extent that $dV^{\delta}/dY^{(\delta,\Delta)}$ differs from dV/dY (and the same remarks as for $Y^{(\delta,\Delta)}$ apply with respect to P). Recall that we define $V^{\circ}\left(Y\right)$ to be the volume of single asset phase space (normalized to $\overline{N}$ and with Y rather than P the functional argument: $V^{\circ}\left(Y\right)=V\left(P\right)$). Therefore, the counterpart risky volume is defined by

$$V^{\circ\delta}\left(Y^{(\delta,\Delta)}\right)=\overline{N}\left(1/\int_{y<\overline{Y}}d\mathfrak{V}^{\delta}\right)\int_{y<Y^{(\delta,\Delta)}}d\mathfrak{V}^{\delta},$$

where we again assume that $Y^{(\delta,\Delta)}$ is bounded above by $\overline{Y}$ (and for simplicity write $\overline{Y}$ for $\overline{Y^{(\delta,\Delta)}}$, $\overline{N}$ for $\overline{N^{(\delta,\Delta)}}\equiv\overline{N+\Delta N}$, and y is the dummy variable corresponding to $Y^{(\delta,\Delta)}$). Since $Y^{(\delta,\Delta)}$ is linear homogenous (in fact, linear) in (P,η), the relation

$$y\left(P,\eta;\cdot\right)<Y^{(\delta,\Delta)}$$

is equivalent to

$$y\left(P/Y^{(\delta,\Delta)},\eta/Y^{(\delta,\Delta)};\cdot\right)<1,$$

and hence, recalling that $d\mathfrak{V}^{\delta}$ is proportional to $dV^{\delta}=dN\,dP\;d\eta_{1}\cdots d\eta_{L}$, $V^{\circ\delta}$ becomes

$$V^{\circ\delta}\left(Y^{(\delta,\Delta)}\right)=$$

$$\overline{N}\left(1/\int_{y<\overline{Y}}d\mathfrak{V}^{\delta}\right)\left(Y^{(\delta,\Delta)}\right)^{1+L}\int_{y<1}d\mathfrak{V}^{\delta}=\frac{\overline{N}}{\overline{Y}^{1+L}}\left(Y^{(\delta,\Delta)}\right)^{1+L},$$

implying that

$$\frac{dV^{\circ\delta}}{dY^{(\delta,\Delta)}}=\frac{(1+L)\,\overline{N}}{\overline{Y}^{1+L}}\left(Y^{(\delta,\Delta)}\right)^{L};$$

that is, the density of states varies as the Lth power of $Y^{(\delta,\Delta)}$, L being the dimensionality of risk in S or B: L equals L_{S} or L_{B}.

Seeing, therefore, that the risky density of states has been established, we arrive at the important formulae:

$$\int_0^{\overline{Y}} \Psi^{(\delta,\Delta)} \frac{dV^{\circ\delta}}{dY^{(\delta,\Delta)}} \, dY^{(\delta,\Delta)} =$$

$$\begin{cases} \dfrac{(1+L_S)\,\overline{N}}{\overline{Y}^{1+L_S}} \displaystyle\int_0^{\overline{Y}} \frac{\left(Y^{(\delta,\Delta)}\right)^{L_S}}{\exp\left[\nu_S^* + \Delta\nu_S^* + Y^{(\delta,\Delta)}/\left(\alpha^* + \Delta\alpha^*\right)\right] + 1} \, dY^{(\delta,\Delta)} & (S) \\[2ex] \dfrac{(1+L_B)\,\overline{N}}{\overline{Y}^{1+L_B}} \displaystyle\int_0^{\overline{Y}} \frac{\left(Y^{(\delta,\Delta)}\right)^{L_B}}{\exp\left[\nu_B^* + \Delta\nu_B^* + Y^{(\delta,\Delta)}/\left(\alpha^* + \Delta\alpha^*\right)\right] - 1} \, dY^{(\delta,\Delta)} & (B) \end{cases}$$

with

$$Y^{(\delta,\Delta)} = (\rho^* + \Delta\rho^*)\, hP + \sum_\lambda \eta_\lambda \, \delta\chi^\lambda.$$

In terms of price:

$$\int_0^{\overline{P}} \Psi^{(\delta,\Delta)} \frac{dV^{\delta}}{dP^{(\delta,\Delta)}} \, dP^{(\delta,\Delta)} = \tag{11}$$

$$\begin{cases} b_S \displaystyle\int_0^{\overline{P}} \frac{\left(P^{(\delta,\Delta)}\right)^{L_S}}{\exp\left[\nu_S^* + \Delta\nu_S^* + (\rho^* + \Delta\rho^*)\, hP^{(\delta,\Delta)}/\left(\alpha^* + \Delta\alpha^*\right)\right] + 1} \, dP^{(\delta,\Delta)} & (S) \\[2ex] b_B \displaystyle\int_0^{\overline{P}} \frac{\left(P^{(\delta,\Delta)}\right)^{L_B}}{\exp\left[\nu_B^* + \Delta\nu_B^* + (\rho^* + \Delta\rho^*)\, hP^{(\delta,\Delta)}/\left(\alpha^* + \Delta\alpha^*\right)\right] - 1} \, dP^{(\delta,\Delta)} & (B) \end{cases}$$

with

$$\begin{aligned} b_S &= (1+L_S)\,\overline{N} \left(\frac{(\rho^* + \Delta\rho^*)\,h}{\overline{Y}}\right)^{1+L_S} \\ b_B &= (1+L_B)\,\overline{N} \left(\frac{(\rho^* + \Delta\rho^*)\,h}{\overline{Y}}\right)^{1+L_B} \end{aligned}$$

and

$$\overline{P} \simeq \overline{Y}/\,(\rho^* + \Delta\rho^*)\,h$$

and

$$P^{(\delta,\Delta)} = P + \frac{1}{(\rho^* + \Delta\rho^*)\,h} \sum_\lambda \eta_\lambda \, \delta\chi^\lambda \simeq P + \sum_\lambda \frac{\partial P}{\partial \chi^\lambda} \, \delta\chi^\lambda.$$

These formulae for distribution of price constitute the principal empirically testable content of this paper.

Observe that $P^{(\delta,\Delta)}$ is the observable price; that L_S and L_B are regarded as fixed integers to be determined empirically; that h is regarded as a constant to be determined empirically; that the constants b premultiplying the integrated distributions are essentially immaterial, because they are absorbed by normalizations; and that the upper limit $\overline{P}$ in the integral becomes immaterial, provided that $\Psi^{(\delta,\Delta)}$ falls to almost zero within the observed range of upper prices. (For the purposes of the price distribution, it seems likely that the upper bound $\overline{P}$ could in many practical cases be approximated by $+\infty$ in view of the factor $\exp\left[(\rho^* + \Delta\rho^*)\, hP^{(\delta,\Delta)}/\,(\alpha^* + \Delta\alpha^*)\right]$ in the denominator of the integrand; for other purposes, however, such as the variation of $\langle N \rangle$ with $(\alpha^* + \Delta\alpha^*)$, the finiteness of $\overline{P}$ may become important.)

Further, observe that

$$(\nu_S^* + \Delta\nu_S^*) \neq (\nu_B^* + \Delta\nu_B^*)$$

(in general), the parameters being determined by arbitrarily given quantities of respectively stocks and bonds; and that, under the assumption of equilibrium in $\mathfrak{F}$, $(\alpha^* + \Delta\alpha^*)$ is common to S and B, and (with the assumption of investor indifference to the kind of asset) $(\rho^* + \Delta\rho^*)$ also is common. Henceforth, we drop the risky suffix: for example, we write $Y^{(\delta,\Delta)} \equiv Y$, $P^{(\delta,\Delta)} \equiv P$.

We shall not attempt in this paper a comparison with empirical data, such as the previously cited Jackwerth and Rubinstein (1996) and Kim (1999). Let us make a few simple observations, however, about the formulae we have derived for the distributions of price.

First, for stocks, observe that for prices P sufficiently high that

$$\exp\left[\nu_S^* + \Delta\nu_S^* + (\rho^* + \Delta\rho^*)\, hP^{(\delta,\Delta)}/\,(\alpha^* + \Delta\alpha^*)\right] \gg 1,$$

the distribution takes the form of a gamma function:

$$\Gamma(n+1) = \int_0^\infty x^n \exp(-x)\; dx \quad (= n!).$$

(This also holds true for all P in the case of the quantity of stocks issued being sufficiently low that $\exp\left[\nu_S^* + \Delta\nu_S^*\right] \gg 1$.) It is known that the gamma function can approximate a lognormal function with the empirically desired differences of a high peak and a fat left (low P) tail; the tail of the exact form of the distribution for $P \to 0$ varies as P^L, just like the gamma function tail, aside from a constant factor $\exp\left[\nu_S^* + \Delta\nu_S^*\right] + 1$. Casual observation suggests that fitted L_S would be relatively small, perhaps $L_S \sim 10^1$.

Second, for bonds, observe that for relatively high values of L_B, perhaps $L_B \sim 10^2$, the distribution will be approximately an increasing exponential, up to a very sharp peak, since x^n behaves approximately exponentially for large n, and $x/\left(\exp[x] - 1\right) \to 0$ as $x \to 0$. Beyond the peak, it will be a sharply declining exponential, varying as

$$\exp - \left[\nu_B^* + \Delta\nu_B^* + (\rho^* + \Delta\rho^*)\, hP^{(\delta,\Delta)}/\,(\alpha^* + \Delta\alpha^*)\right].$$

That is, for large L_B the bond price distribution matches *prima facie* the empirical data on the distribution arising from credit risk, provided that we interpret short-run changes in credit risk (e.g. quarterly) as realizations of expectations.

Third, for stocks and bonds, we can estimate the order of magnitude of

$$(\nu_S^* + \Delta\nu_S^*)$$

and L_B from the empirical widths of the price distributions. To see this, consider the theoretical upper halfwidth of each distribution, defined as the difference in price $\Delta P > 0$ from the peak at price P to a value at $P + \Delta P$ that is much smaller than the peak. For stocks, considering the case of the quantity of stocks issued sufficiently large that $\exp[\nu_S^* + \Delta\nu_S^*] \to 0$ (implying $[\nu_S^* + \Delta\nu_S^*] < 0$), the peak occurs at around a value of P given by

$$(\rho^* + \Delta\rho^*)\, hP = -\,(\alpha^* + \Delta\alpha^*)\,(\nu_S^* + \Delta\nu_S^*)\,,$$

since the factor $1/\left\{\exp\left[\nu_S^* + \Delta\nu_S^* + (\rho^* + \Delta\rho^*)\, hP/\,(\alpha^* + \Delta\alpha^*)\right] + 1\right\}$ falls rapidly from 1 to $0+$ around that value. The halfwidth is of the order of the inverse quantity multiplying P in that distribution: $\Delta P \sim (\alpha^* + \Delta\alpha^*)\,/\,(\rho^* + \Delta\rho^*)\, h$. Hence

$$\frac{\Delta P}{P} \sim \frac{1}{|(\nu_S^* + \Delta\nu_S^*)|},$$

as to order of magnitude. Empirically, we find a halfwidth of roughly 10 percent: $\Delta P/P \approx 0.1$ (Jackwerth and Rubinstein, 1996), implying that $(\nu_S^* + \Delta\nu_S^*) \sim -10$. For bonds, presuming that $\exp\left[\nu_B^* + \Delta\nu_B^* + (\rho^* + \Delta\rho^*)\, hP^{(\delta,\Delta)}/\,(\alpha^* + \Delta\alpha^*)\right] \gg 1$ at the turning point (since $(\nu_B^* + \Delta\nu_B^*) \geq 0$), we see that at that point $P = L_B\,(\alpha^* + \Delta\alpha^*)\,/\,(\rho^* + \Delta\rho^*)\, h$, and hence

$$\frac{\Delta P}{P} \sim \frac{1}{L_B}.$$

Since empirically, the upper halfwidth is roughly 1 percent, $\Delta P/P \approx 0.01$ (Kim, 1999), we have $L_B \sim 10^2$. (A note of caution is in order here for stocks. We have assumed that $\exp[\nu_S^* + \Delta\nu_S^*] \to 0$; if, to the contrary, one considered the case of a small quantity of stocks, $\exp[\nu_S^* + \Delta\nu_S^*] \to +\infty$, then an argument analogous to that for bonds would lead to $\Delta P/P \sim 1/L_S$, giving the result $L_S \sim 10^1$. To distinguish the two cases of $[\nu_S^* + \Delta\nu_S^*]$ would require careful empirical comparison, or other theoretical arguments.)

Remark 2 *The theory holds independently of some commonly invoked special assumptions such as special forms for the utility functions (contrast the treatment in the Appendix); the existence of a representative agent—to the contrary, agents are heterogeneous in behaviour; zero net issuance of new securities or an arbitrary assignment of firms' cashflows between net issuance and dividends; and an exogenous (more specifically, fixed) interest rate.*

Let us turn now to a financial interpretation of the theory, that is, an interpretation of the changes in the macrofinancial parameters: α^*, $\langle N\rangle$ (or the counterpart ν^*), and the χ. We shall find that we can decompose changes in $\langle Y\rangle$ (but not Y, which is a random quantity) into changes attributable to market fundamentals, investor sentiment, and investor acquisition of securities. Since the relation will prove to be a differential one, we henceforth consider only infinitesimal variations in the macrofinancial parameters: $d\chi$, $d\alpha^*$, and $d\langle N\rangle$ (and the counterpart $d\nu^*$), and the market considered will henceforth be the aggregate market $\mathfrak{F} = S + B$. We shall incidentally provide some more rigorous underpinnings of the preceding results.

Consider first, a change in market fundamentals $d\chi$ from a prevailing value χ. Then the effect of $d\chi$ on $\mathfrak{F} = S + B$ is determined in terms of the observable $(\widehat{\partial P/\partial\chi^\lambda})_\kappa$ and $\widehat{\partial\rho^*/\partial\chi^\lambda}$, as can be seen from the following considerations. It is convenient to conduct the argument in discrete terms, momentarily reverting to the notation of the earlier part of Section 3, and to consider the component factors of $Y = \rho^* hP$. In just the same way as for Φ and $\langle N^\iota\rangle$, define the corresponding quantities for each component by asset type $\{\kappa\}$, that is, the generating function ϕ_κ and the average (expected mean) $\langle N^{\iota\kappa}\rangle_\kappa$, excepting that the averages pertain only to the κ phase space and $\langle\ \rangle_\kappa$ denotes an average over that phase space.

Consider the relationship of $\langle N^\iota\rangle$ to $\langle N^{\iota\kappa}\rangle$. Since $N^\iota = \sum_\kappa N^{\iota\kappa}$, we have $\langle N^\iota\rangle = \sum_\kappa \langle N^{\iota\kappa}\rangle$ (by the theorem that the mean value of the sum equals the sum of the mean values, regardless of whether or not the random quantities are independent). Further, since market $Y = \sum Y_\kappa$ and the component Y_κ can be expressed as functions of their respective phase space coordinates alone, we can equate $\langle N^{\iota\kappa}\rangle = \langle N^{\iota\kappa}\rangle_\kappa$, and hence $\langle N^\iota\rangle = \sum_\kappa \langle N^{\iota\kappa}\rangle_\kappa$. Consequently, by (2), $\partial \log\Phi/\partial P_\iota = \sum_\kappa \partial\log\phi_\kappa/\partial P_\iota$. By (3) and a parallel argument applied to $Y = \sum Y_\kappa$, we find that $\partial\log\Phi/\partial\rho^* = \sum_\kappa \partial\log\phi_\kappa/\partial\rho^*$.

Hence,

$$
\begin{aligned}
\frac{\partial\log\Phi}{\partial\chi^\lambda} &= \sum_\iota \left\{ \frac{\partial\log\Phi}{\partial P_\iota}\left(\frac{\partial P_\iota}{\partial\chi^\lambda}\right)_\kappa + \frac{\partial\log\Phi}{\partial\rho^*}\left(\frac{\partial\rho^*}{\partial\chi^\lambda}\right)\right\} \\
&= \sum_\iota\sum_\kappa \left\{ \frac{\partial\log\phi_\kappa}{\partial P_\iota}\left(\frac{\partial P_\iota}{\partial\chi^\lambda}\right) + \frac{\partial\log\phi_\kappa}{\partial\rho^*}\left(\frac{\partial\rho^*}{\partial\chi^\lambda}\right)\right\} \\
&= -\frac{\rho^* h}{\alpha^*}\sum_\kappa \left\langle\frac{\partial P}{\partial\chi^\lambda}\right\rangle_\kappa - \frac{h}{\alpha^*}\frac{\partial\rho^*}{\partial\chi^\lambda}\sum_\kappa \langle P\rangle_\kappa \\
&= -\frac{1}{\alpha^*}\sum_\kappa \left\langle\frac{\partial Y_\kappa}{\partial\chi^\lambda}\right\rangle_\kappa \\
&= -\frac{1}{\alpha^*}\left\langle\frac{\partial Y}{\partial\chi^\lambda}\right\rangle \qquad\qquad (\lambda = 1,\ldots,L)\,.
\end{aligned}
$$

The third line of the above set shows that $\partial\log\Phi/\partial\chi^\lambda$ is determined in terms of the observable $(\widehat{\partial P/\partial\chi^\lambda})_\kappa$ and $\widehat{\partial\rho^*/\partial\chi^\lambda}$, since we always identify phase space averages with observed macrofinancial quantities: $\langle\partial P/\partial\chi^\lambda\rangle_\kappa = (\widehat{\partial P/\partial\chi^\lambda})_\kappa$ and likewise for ρ^*. From the last line, we can write

$$dW = \sum_{\lambda} \left\langle \frac{\partial Y}{\partial \chi^{\lambda}} \right\rangle d\chi^{\lambda} = -\alpha^{*} \sum_{\lambda} \frac{\partial \log \Phi}{\partial \chi^{\lambda}} d\chi^{\lambda},$$

where dW expresses the change in $\langle Y \rangle$ attributable to a change in market fundamentals $d\chi$. It is clear that we refer only to small changes, and that the meaning of small is the validity of a Taylor expansion to first order for $\log \Phi$ (a necessary condition for the permissibility of such an expansion being that the functions that appear in Φ are analytic, as noted in Section 2). Underlying this expression for dW is the assumption that markets are frictionless and, in particular, that there are no information costs in observing the χ.

Consider second, a given change in cash allocations to $\mathfrak{F}$ from $\mathfrak{F}^{*}$ (with fluctuations in real economy variables zero: $d\chi = 0$). As stated, this intuitive notion is not well defined, since investors who buy are matched by investors (or market making intermediaries) who sell. We can arrive at a consistent definition of the financial flow in terms of the change in market value in the following way. Think of $\mathfrak{F} \subset \mathfrak{E}$ as a national economy comprising a fixed set of residents $\{\mu\}$; then $\langle Y \rangle$ can be thought of as the financial asset component of national income. $\mathfrak{F}^{*} \subset \mathfrak{E}^{*}$ is regarded as the world economy.

Consider a non-zero fluctuation $\langle Y \rangle \rightarrow \langle Y \rangle + d\langle Y \rangle$. Setting aside for the moment any capital account transactions, $d\langle Y \rangle$ has a counterpart in $\mathfrak{E}$ in a fluctuation in the current account surplus CA of the same magnitude and sign:

$$d\langle CA \rangle = d\langle Y \rangle .$$

For example, we can think of $d\langle CA \rangle$ as resulting in an accumulation of currency on the balance sheets of the firms in $\mathfrak{E}$, leading to an equal increase in the market value of firms, $d\langle Y \rangle$, referring to aggregate $\mathfrak{F} = S + B$ (recall that in $\mathfrak{E}^{*}$ there are no foreign currencies and only private agents). Stated more generally, to changes in the current account there correspond changes in national income, of which a component is financial asset income. In this way, we can associate changes in the value of financial assets arising from external financial flows with a notional component of the current account, which we shall call the financial asset component of the current account, denoted by dX in the infinitesimal case. Formally, we define dX to be that part of $d\langle Y \rangle$ that is not attributable to changes in fundamentals:

$$\begin{aligned} dX &= d\langle Y \rangle - dW \qquad by\ definition \\ &= d\langle Y \rangle + \alpha^{*} \sum_{\lambda} \frac{\partial \log \Phi}{\partial \chi^{\lambda}} d\chi^{\lambda}, \end{aligned}$$

in the case of a closed capital account, consistent with the notion that current accounts plus currency flows sum to zero. Further, to the extent that, in particular instances, one can regard changes in valuations $(d\langle Y \rangle - dW)$ as driving fluctuations in the current account, one can identify dX with investor sentiment. In terms of the macrofinancial parameters, a change $d\langle Y \rangle$

with a non-zero component $dX \neq 0$ is reflected in a corresponding change $d\alpha^*$, which shifts the probability distribution Ψ from lower to higher Y if $d\alpha^* > 0$, and conversely. (We deliberately ignore the possibility of $d\langle Y\rangle$ arising from domestic causes in $\mathfrak{C}$, for instance a technology shock, which are not the subject of our present inquiry.)

Considering at last an open capital account for $\mathfrak{C}$, a change $d\langle Y\rangle$ arising from a change $d\langle N\rangle$ in $\langle N\rangle$ signifies the acquisition of assets abroad. Residents $\{\mu\}$ trade with non-residents $\notin \{\mu\}$ the set of assets of given types $\{\kappa\}$ (—for simplicity, we exclude the possibility that the types $\{\kappa\}$ might be enlarged).

These considerations lead us to introduce the fluctuations $d\alpha^*$ and $d\langle N\rangle$, in addition to $d\chi$. The consequent change $d\log\Phi$ in $\log\Phi$ is found by a first order expansion of (10) to be the expression

$$d\log\Phi = \sum_{\lambda}\frac{\partial\log\Phi}{\partial\chi^{\lambda}}\,d\chi^{\lambda} + \frac{1}{\alpha^{*2}}\,\langle Y\rangle\,d\alpha^* + \nu^*\,d\langle N\rangle\,. \tag{12}$$

Let us first clarify the relationship between $\langle N\rangle$ and ν^*, so as to characterize Ψ accurately (although in fact we shall not be concerned with Ψ explicitly in what follows). Note that $\log\Phi$ is independent of ν^*, $\partial\log\Phi/\partial\nu^* = 0$, which follows from (10) and (8); equivalently, we may say that $\log\Phi\left(\langle N\rangle\,;\cdot\right)$ and $\left(\log\Phi - \langle N\rangle\,\nu^*\right)$ are reciprocally Legendre transforms the one of the other. This means not that ν^* is superfluous, but rather that it is an alternative (a dual variable) to $\langle N\rangle$, and reciprocally the one determines the other; that is, $\nu^* = \nu^*\left(\langle N\rangle\right)$, and $\langle N\rangle + \Delta\langle N\rangle$ determines a corresponding $\nu^* + \Delta\nu^*$ through (8). Accordingly, we replace ν^* by $\nu^* + \Delta\nu^*$ in Ψ. The counterpart to $\Delta\langle N\rangle$ is given by the $\Delta\nu^*$ term in Ψ (stated more precisely, we require ν^* to satisfy $-\partial\left(\log\Phi - \langle N\rangle\,\nu^*\right)/\partial\nu^* = \langle N\rangle + \Delta\langle N\rangle$).

Since we may regard $\langle Y\rangle$ as $\langle Y\rangle\left(\alpha^*, \nu^*, \chi\right)$ in view of the form of the distribution Ψ, we could expand $d\langle Y\rangle$ in the usual way by the first order differential form

$$d\langle Y\rangle = \frac{\partial\langle Y\rangle}{\partial\alpha^*}\,d\alpha^* + \frac{\partial\langle Y\rangle}{\partial\nu^*}\,d\nu^* + \sum_{\lambda}\frac{\partial\langle Y\rangle}{\partial\chi^{\lambda}}\,d\chi^{\lambda}.$$

Superficially, one might suppose that this form provides a basis for decomposing $d\langle Y\rangle$ into separate elements: for instance, associating the first term with a change in current account flows. But, in fact, the expression conjures up a mathematical illusion, for we do not know whether, for instance, $\left(\partial\langle Y\rangle/\partial\alpha^*\right)d\alpha^*$ can be expressed as a differential of a function of the state variables (the observable macrofinancial parameters α^*, $\langle N\rangle$, χ^{λ}, or their duals, $\langle Y\rangle$, ν^*, $(1/\alpha^*)\,\eta_{\lambda}$), or whether, to the contrary, a finite change $\int\left(\partial\langle Y\rangle/\partial\alpha^*\right)d\alpha^*$ is path dependent. Put another way, we have no assurance that a zero current account flow corresponds to $d\alpha^* = 0$, or even that there exists any such correspondence $df = 0$, where f is a function of the state variables.

In mathematical terms, the problem here (with $2 + L > 2$) is essentially the same as the problem of integrability in utility theory (with the number of goods in the consumption bundle

> 2). The Pfaffian (the first order differential form) cannot be assumed *a priori* to be an exact differential (that is, a differential of a function of the state variables), unless its coefficients satisfy certain reciprocal integrability conditions. (And it seems that it may be possible to apply the probabilistic approach of this paper to the theory of consumer's preference, treating the quantities of goods as random variables, so as to establish integrability and to construct a utility function, in a manner that bears some similarity to von-Neumann-Morgenstern utility.)

By use of the expression for $d \log \Phi$, however, we can construct a satisfactory decomposition (one that establishes integrability). From (12) it follows that

$$\begin{aligned}
& d\langle Y\rangle - dW \\
= \;& d\langle Y\rangle + \alpha^* \sum_\lambda \frac{\partial \log \Phi}{\partial \chi^\lambda} d\chi^\lambda \\
= \;& d\langle Y\rangle + \alpha^* \left(d\log\Phi - \frac{1}{\alpha^{*2}} \langle Y\rangle \, d\alpha^* - \nu^* d\langle N\rangle \right) \\
= \;& \alpha^* \left(\frac{1}{\alpha^*} d\langle Y\rangle - \frac{1}{\alpha^{*2}} \langle Y\rangle \, d\alpha^* + d\log\Phi \right) - \alpha^* \nu^* d\langle N\rangle \\
= \;& \alpha^* d\left(\frac{1}{\alpha^*} \langle Y\rangle + \log\Phi \right) - \alpha^* \nu^* d\langle N\rangle
\end{aligned}$$

Hence

$$\begin{aligned}
d\langle Y\rangle &= -\alpha^* \sum_\lambda \frac{\partial \log \Phi}{\partial \chi^\lambda} d\chi^\lambda + \alpha^* d\left(\frac{1}{\alpha^*} \langle Y\rangle + \log\Phi \right) - \alpha^* \nu^* d\langle N\rangle \\
&= \sum_\lambda \eta_\lambda \, d\chi^\lambda + \alpha^* d\left(\frac{1}{\alpha^*} \langle Y\rangle + \log\Phi \right) - \alpha^* \nu^* d\langle N\rangle
\end{aligned}$$

and the exact differential

$$d\left(\frac{1}{\alpha^*} \langle Y\rangle + \log\Phi \right)$$

may be equated to $(1/\alpha^*)\, dX$, by noting that no change in market fundamentals and a closed capital account correspond to $d\chi = 0$, $d\langle N\rangle = 0$ (again, with the underlying assumption of frictionless markets).

Accordingly, we may write the important formula

$$d\langle Y\rangle = dW + dX + dZ \tag{13}$$

where

$$
\begin{aligned}
dW &= \sum_{\lambda} \eta_{\lambda}\, d\chi^{\lambda} \\
dX &= \alpha^* d\left(\frac{1}{\alpha^*}\langle Y\rangle + \log\Phi\right) \\
dZ &= -\alpha^* \nu^*\, d\langle N\rangle,
\end{aligned}
\tag{14}
$$

and interpret change $d\langle Y\rangle$ as attributable to changes dW arising from market fundamentals, dX from investor sentiment, and dZ from investor acquisition of securities.

Very briefly, let us sketch out an argument as to how the formulae (11) suggest that small random cash allocations to the market could generate a logarithmic Wiener price process through time. Throughout this paper, we have been concerned with momentary equilibrium at a point in time. Consider now a small fluctuation $\delta\alpha^*(t)$ that varies through time with, for simplicity, a mean of zero.

A fluctuation $\delta\alpha^*$ can be viewed as a change $\Delta\langle Y\rangle$ in the expected portfolio value (of S or B), arising from random allocations of cashflow between $\mathfrak{F}^*$ and $\mathfrak{F}$. (Such random allocations should be viewed not as deviations from equilibrium, but as components of an equilibrium that is dynamic, according to the work of this paper.) The change $\Delta\langle Y\rangle$ will be reflected in a changed expectation of capital gain in price. From (11), the equivalent fluctuation in price is given by

$$
\Delta P = -\frac{\delta\alpha^*}{\alpha^*}P \qquad \left(\frac{\delta\alpha^*}{\alpha^*} \ll 1\right).
$$

The deterministic Euler-Lagrange optimal path condition is (from the Appendix, using the transformation from current to total return) $dp_{\kappa}/dt = +\rho p_{\kappa}$. Dropping the κ, and writing P for price and ρ^* for ρ, the stochastic version of the condition becomes

$$
\begin{aligned}
\frac{dP}{P} &= \rho^*\, dt - \frac{\delta\alpha^*(t)}{\alpha^*} \\
&= \rho^*\, dt - dB(t),
\end{aligned}
$$

where we write $dB = \delta\alpha^*/\alpha^*$ (purely for notational simplicity). We see that the factor $1/P$ naturally occurs, that is, the random disturbance B can be written independently of P, as a consequence of (11). This factor ensures that the presence of B does not contravene limited liability. Additionally, it is suggestive of a logarithmic price process, which may arise in the following way.

Suppose that $B(t)$ is a stochastic diffusion (or stochastic differential) process, that is, a stationary Markov process for which the

$$
B(t_2) - B(t_1), \cdots, B(t_n) - B(t_{n-1}) \qquad (t_1 < t_2 < \cdots < t_{n-1} < t_n)
$$

are mutually independent random variables. Then, if our assumption that $\delta\alpha^*/\alpha^* \ll 1$ is to hold true, it must be the case that the distribution of B has a finite second moment. For, the equilibrium distribution $\Psi(P)$ for $\mathfrak{F}$ is characterized by the given α^*, with $\langle\delta\alpha^*\rangle = 0$ by assumption, and with the dynamic equilibrium constituting only small deviations from Ψ_0, the distribution with fixed α^*. For Ψ_0, because it satisfies (11), the second (central) moment is given by $\partial^2 \log \Phi(\alpha^*) / \partial (1/\alpha^*)^2$. Hence, the second moment of Ψ is approximately equal to $\partial^2 \log \Phi(\alpha^*) / \partial (1/\alpha^*)^2$ (that is, neglecting the $\delta\alpha^*$) and that quantity is finite. Then, the corresponding moment of the distribution for α^* must also be finite, since in equilibrium, P and $(1/\alpha^*)$ are dual variables, and their respective distribution second moments are reciprocals one of the other, given appropriate normalizing constants. (To see this, observe that the dual distribution for $(1/\alpha^*)$ can be derived from the Legendre transform of $\log \Phi(\alpha^*; \cdot)$ with respect to the variable α^*, to be distinguished from the transform with respect to $\langle N\rangle$ discussed earlier.)

With a finite second moment, and excluding by assumption the possibility of jumps in $B(t)$, that is, assuming $B(t)$ is continuous with probability 1, it follows that $B(t)$ is Gaussian, with a variance $\sigma^2(t)$ that is proportional to t: $\sigma^2(t) = t\sigma^2$—see, for instance, Doob (1942), sect. 3. With the assumption that the drift coefficient (the interest rate) $\rho^*(t)$ is slowly varying, we see that dP/P follows a Gaussian distribution, and P follows a logarithmic Wiener process. (More generally, if the second moment of B is not finite, then B, and consequently P, will follow a general stochastic diffusion process—Doob (1942), sect. 5.)

Finally, let us observe that the methodology of this paper has expressed in an essential way the character of assets, namely real or fiat. It seems reasonable to suppose, therefore, that it could be carried over, *mutatis mutandis*, to other applications: for example, to houses as real assets and to their distribution by price; to insurance contracts as fiat assets and to their distribution by net income; and, more generally, to derivatives pricing.

APPENDIX I

Given the $\mathfrak{F}$ defined in section 2, assume that investors are infinitely long lived and supply labor l^{μ} inelastically, with the time path of wages w_{μ} exogenous. We shall always consider the case where n is bounded by arbitrarily given functions, $\underline{n}(t) \leq n(t) \leq \overline{n}(t)$, which may be interpreted as creditworthiness constraints on exposure to counterparties. Stipulate that the optimal path $O^{\oplus}_{A\Omega} \equiv n^{\oplus}(t)$ (O from $o\delta o\varsigma$, Greek for path or way) lies on the interior of the bounded state space $\mathfrak{U}$. Within this framework, we shall find that the microfinancial state is indeterminate, in that $dn^{\oplus}/dt$ and $n^{\oplus}(t)$ are indeterminate, that is, may take any of a KM–fold infinity of solutions; nevertheless, the subsequent development in section 2 shows that the macrofinancial state is determinate.

So as to highlight the Lagrangian of the problem and the significance of its invariance, let us apply the formalism of the calculus of variations. (Elsewhere (Johannes, 2001), I have derived similar results in a Hamiltonian formalism, by use of the Pontryagin maximum principle.) We shall derive purely necessary conditions and assume that at least one solution $O^{\oplus}_{A\Omega}$ exists. Further, we shall assume, without rigorous argument, that at points of finite dividend or interest payment, jumps occur in the optimal path $n^{\oplus}(t)$ in $\mathfrak{U}$; and that, in consequence, the transversality conditions become redundant, thereby justifying the assignment of initial $c(t_A)$, $p(t_A)$, $p_{-1}(t_A)$ as given empirical parameters.

An individual consumer maximizes utility flows:

$$\max_{\dot{c}(\tau),\dot{n}(\tau)} \int_{t_A}^{t_\Omega} d\tau \exp\left(-\int_{t_A}^{\tau} \theta\, d\tau'\right) U(c),$$

subject to the budget constraint

$$p_\kappa \frac{dn^\kappa}{dt} h = d_\kappa n^\kappa h - p_{-1} c + wl,$$

(where we adopt the summation convention that there is an implied summation over variables appearing as both covariant and contravariant indices in a single term, for example, $p_{\kappa\mu} n^{\kappa\mu} \equiv \sum_\kappa \sum_\mu p_{\kappa\mu} n^{\kappa\mu}$, and provisionally suppressing index μ) through choice of the $(1+K)$ controls $\dot{c}(\tau), \dot{n}(\tau)$, where we regard c as a state variable and $\dot{c} \equiv dc/dt$ as a control variable; and subject also to the initial condition $n(t_A) = n_A$; and the bounds $c(t) \geq 0$ and $\underline{n}(t) < n(t) < \overline{n}(t)$, where short selling is limited by the creditworthiness constraints. To ensure an interior solution $c > 0$, we make the traditional assumptions: $dU/dc > 0$; $d^2U/dc^2 < 0$; $\lim_{c\to 0} dU/dc = \infty$; $\lim_{c\to\infty} dU/dc = 0$; and further require $c(t)$ to be piecewise continuous.

The Lagrangian $\mathfrak{L}$ for the problem may be written

$$\begin{aligned}\mathfrak{L} &= \mathfrak{L}\left(\tau, c, n, \dot{n}\right) \\ &= \exp\left(-\int_{t_A}^{\tau} \theta \, d\tau'\right) U(c) \\ &\quad + \exp\left(-\int_{t_A}^{\tau} \theta \, d\tau'\right) \lambda \left(d_\kappa n^\kappa h - p_\kappa \frac{dn^\kappa}{dt} h - p_{-1} c + wl\right),\end{aligned}$$

where the investor maximizes $\int \mathfrak{L} \, d\tau$ through unconstrained control variables (and $\dot{c}$ is an implicit control variable). It is convenient to consider also the sub-Lagrangian (or simply "Lagrangian") L in the subspace $\left(\tau, n, \dot{n}\right)$

$$L\left(\tau, n, \dot{n}\right) = \exp\left(-\int_{t_A}^{\tau} \theta \, d\tau'\right) \lambda \left(d_\kappa n^\kappa h - p_\kappa \frac{dn^\kappa}{dt} h\right)$$

where implicitly $c = c^{\oplus}$ its optimized value.

Without attempting a rigorous treatment, let us note first, that the applicability of the classical calculus of variations requires the control space to be either open or equal to all of $R^{(1+K)M}$, the latter requirement being fulfilled here since we assume $dc/dt \in (-\infty, +\infty)$ and $dn^\kappa/dt \in (-\infty, +\infty)$, consistent with the admissibility of piecewise continuous $c(t)$ and potential finite trading of n. Second, that the class of admissible solutions $n^{\oplus}(t)$ is the absolutely continuous functions.

At points of discontinuity in d_κ, the calculus of variations is not applicable, and consequently $O^{\oplus}_{A\Omega}$ on $[t_A, t_\Omega]$ (where we let $t_\Omega \to \infty$) is decomposed into an endless sequence of finite arcs: $[t_\alpha, t_\beta]$, $[t_\beta, t_\gamma]$, $\cdots$, to each of which the formalism can be applied; and limits in section 2 should be so understood. (It is possible to apply a finite form of the Euler-Lagrange equations so as to show that, for instance, price jumps downward by the amount of the dividend in the absence of tax; it was Lagrange himself who developed such a form. But the state variables are not absolutely continuous across the discontinuity.) Here, d_κ may denote, for instance, a Dirac delta improper function, but for simplicity of argument, we keep the functional form implicit. At points of finite cashflow, given that $c(t)$ is piecewise continuous, $n^{\oplus}$ necessarily undergoes a finite jump (almost always), since finite savings must be allocated to the securities market (ignoring singular cases, such as savings across investors proportionate to n^κ being reinvested so that p_κ remains absolutely continous). We see also that at points where c is not absolutely continuous there will be breaks in $O^{\oplus}_{A\Omega}$. Similar arguments could be made in respect of finite wage payments, finite issuance of securities, and finite stochastic shocks.

Since finite trading in the market by each investor may lead to any portfolio of equal value, that value being determined by the evolution of ρ, the transversality conditions at the endpoint of each arc become redundant. That is, a succession of transversality conditions extending to $t_\Omega \to \infty$ does not constrain the optimal path by backward induction of investor expectations, because of the possibility of finite trading; and investors need exercise only short-run rational expectations, as we have assumed. Consequently, along with $n(t_A)$, we regard $p(t_A)$, $p_{-1}(t_A)$, $c(t_A)$ as given empirical parameters.

On a finite arc, the Euler–Lagrange equations give:

$$\frac{dU}{dc} - \lambda p_{-1} = 0 \tag{A-1}$$

$$\frac{dp_\kappa}{dt} = -d_\kappa + \theta p_\kappa - \frac{1}{\lambda}\frac{d\lambda}{dt}p_\kappa \qquad (\kappa = 1, \ldots, K). \tag{A-2}$$

Since the money market deposit is the numeraire—embodying the economic assumption that money is traded in a single market at a single price by all investors (that is, each investor trades money with every other investor, either directly or indirectly)—A-2 imply

$$\lambda = \lambda_0 \exp\left(\int_{t_\alpha}^{t} (\theta - \rho)\, d\tau\right);$$

$$\frac{dp_\kappa}{dt} = -d_\kappa + \rho p_\kappa \qquad (\kappa = 1, \ldots, K),$$

where $\lambda_0 > 0$ (by A-1) is constant, and $p(t)$ is independent of the unobservable θ; note that there is a purely formal similarity to stochastic discount factors and transformation to the risk-neutral basis. (Here, it seems possible that $p_\kappa = 0$ in a finite time when $d_\kappa > \rho$. We shall find, however, that in the risky equilibrium of section 4, the probability of occurrence of assets bearing zero price becomes vanishingly small. That is, the macroeconomic equilibrium ensures that the probability of occurrence of dynamically inefficient microeconomic states is zero.)

From A-1 we can infer the usual Keynes-Ramsey type rule:

$$\frac{d^2U/dc^2}{dU/dc}\frac{dc}{dt} = \frac{d\log(dU/dc)}{dt} = (\theta - \rho),$$

where we adopt what seems to be the customary assumption that p_{-1} is regarded as a constant. Underlying this assumption is the notion that c considered as a marginal asset offers a zero current return (since by A-1 on $O^{\oplus}_{A\Omega}$ the utility benefit is simultaneously offset by the financial cost at the margin), and zero total return (since the asset is extinguished after consumption); and implicit is the economic supposition that consumers hold c for the purpose of consumption only, not for trading, and there are no commodity markets for consumption goods.

Consequently,

$$c^{\oplus}(t) = [dU/dc]^{-1}\left[\frac{dU(c(t_\alpha))}{dc}\exp\int_{t_\alpha}^{t}(\theta - \rho)\, d\tau\right]$$

where $c(t_\alpha)$ is assumed given, and $[dU/dc]^{-1}$ is the inverse mapping of dU/dc (for any cardinal representation); and with determined $c^{\oplus}$, we can work in the subspace $\left(\tau, n, \dot{n}\right)$ of L.

The Weierstrass condition gives:

$$\begin{aligned}&\lambda\left(d_\kappa\, n^{\kappa\oplus}\, h - p_\kappa\,\frac{dn^\kappa}{dt}\,h + p_\kappa^\oplus\,\frac{dn^\kappa}{dt}\,h\right)\\ \le\ &\lambda\left(d_\kappa\, n^{\kappa\oplus}\, h - p_\kappa^\oplus\,\frac{dn^{\kappa\oplus}}{dt} h + p_\kappa^\oplus\,\frac{dn^{\kappa\oplus}}{dt}\,h\right),\end{aligned}$$

where $p^\oplus$ denotes $(1/h)\,\partial L\left(\tau, n^\oplus, \dot{n}^\oplus\right)/\partial\dot{n}$ and again $\oplus$ denotes value on the optimal path $O^\oplus_{A\Omega}$. It follows that for optimality $(p_\kappa^\oplus - p_\kappa)\ dn^\kappa/dt$ is maximized with respect to the $dn^\kappa/dt \in (-\infty, +\infty)$, and hence

$$p_\kappa^\oplus - p_\kappa = 0 \qquad (\kappa = 1, \ldots, K)\,,$$

that is, shadow price equals market price. Then $dn^{\kappa\oplus}/dt$ is indeterminate, that is, can take any of its admissible values $(-\infty, +\infty)$. Strictly speaking, in this equation, as in others, the condition only holds almost everywhere; the point of principle is that with an optimality criterion framed in terms of an integral (of utility flows), one cannot meaningfully distinguish solutions more finely than almost everywhere.

The budget constraint is recovered from the equation for the Lagrange multiplier:

$$d_\kappa\, n^\kappa\, h - p_\kappa\,\frac{dn^\kappa}{dt}\,h - p_{-1}\,c^\oplus + wl = 0.$$

For financial assets, the Legendre conditions are always weakly fulfilled, since the matrix $\left[\partial^2 L/\partial\dot{n}^k\,\partial\dot{n}^l\right]$ is everywhere singular, corresponding to a semi-definite quadratic form. The economic interpretation is that financial assets are characterized by a determinate-price equilibrium, with $dn^\oplus/dt$ nowhere determinate; consequently, in $\mathfrak{F}$, price eigenstates (states of definite p) are equivalent to value eigenstates (states of definite momentary rate of return), as are their distributions.

Aggregating over investors in $\left(t, n, \dot{n}\right)$, we can write (with $d_\kappa \equiv d_{\kappa\mu}$)

$$L = \lambda_0\,\exp\left(-\int_{t_\alpha}^{t}\rho\,d\tau\right)\left(d_{\kappa\mu}\,n^{\kappa\mu}\,h - p_{\kappa\mu}\,\frac{dn^{\kappa\mu}}{dt}\,h\right).$$

The aggregate budget constraint for investors implies that the time path of

$$(p_{\kappa\mu}\,dn^{\kappa\mu}/dt - d_{\kappa\mu}\,n^{\kappa\mu})$$

is determined by $(w_\mu l^\mu - p_{-1\mu}\,c^\mu)$ (given $c^\mu\,(t_A)$), with the composition of aggregate firms' cashflow undetermined as between higher securities' issuance and lower securities' payouts (or *vice versa*). If we assigned an arbitrary rule for the mode of firms' corporate finance, for instance, that securities' payouts were zero (say, no bonds issued and dividend payments effected by share repurchase), or at the opposite pole that securities' net issuance was zero, then the path of

$dn^{\oplus}/dt$ and $n^{\oplus}(t)$ could be fixed. But with such constraints, optimal investor behaviour implies that $p(t) \equiv p^{\oplus}(t)$ would be affected. That is, the price process $p^{\oplus}(t)$ would be disturbed by exogenous fixing of $dn^{\oplus}/dt$ and $n^{\oplus}(t)$. Conversely, postulating a price process $p^{\oplus}(t)$ implies the processes $dn^{\oplus}/dt$ and $n^{\oplus}(t)$, which cannot be assigned independently (and similar remarks apply to a postulated process for dividends). Neither price nor quantity process is assigned in the present theory: $dn^{\oplus}/dt$ and $n^{\oplus}(t)$ remain indeterminate for any given microfinancial state; and $p(t)$ evolves with d_{κ} to give a total return ρ, which is assumed to exist a.e.. We shall see (section 2), however, that the expected aggregate quantities of stocks and bonds and their distribution by price are determinate in a macrofinancial state, characterized by known macrofinancial parameters at a point in time.

Transforming L to present value prices and returns, denoted by a $'$, and setting $\lambda_0 = 1$:

$$L' = d'_{\kappa\mu}\, n^{\kappa\mu}\, h - p'_{\kappa\mu} \frac{dn^{\kappa\mu}}{dt}\, h$$

and $n' = n$. In considering transformations back and forth between current and present value prices at specified discount rates as allowable, it is implicit that L' continues to satisfy the Euler-Lagrange equations, in general form

$$\frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{n}^{\kappa\mu}}\right) - \frac{\partial L'}{\partial n^{\kappa\mu}} = 0 \qquad \kappa = 1, \ldots, K;\ \mu = 1, \ldots, M, \tag{A-3}$$

as can be verified explicitly. Underlying that presumption is the assumption that L is an invariant, that is, a scalar in respect of its independent variables, with unchanging value under a transformation of the coordinate system of those variables. For, it is known that the Euler-Lagrange equations are equivalent to Hamilton's principle $\delta \int L\, dt = 0$ (for variations from the optimal path with fixed endpoints), and so invariance of L is equivalent to covariance of (A-3). Mathematically, L is scalar, and (A-3) define the components of a covariant vector.

In fact, the Euler-Lagrange equations (and all the necessary conditions) are covariant under any transformation of the kind (Cesari, 1983, sect. 2.4):

$$n'^{\kappa\mu} = n'^{\kappa\mu}(t, n) \qquad \kappa = 1, \ldots, K;\ \mu = 1, \ldots, M, \tag{A-4}$$

corresponding to, in general, not only a different coordinate framework for financial assets, but also a different degree of rental (borrowing or lending) of financial assets. The economic interpretation is that not only is the choice of coordinate framework for employed assets a matter of indifference, but so also is the distinction between ownership and rental for employed assets—itself a choice of coordinate framework—since the difference between ρ and $d_{\kappa\mu}$ is accounted for by capital gains.

A consequence of (A-3) is that

$$\frac{d}{dt}\left(L' - \frac{\partial L'}{\partial \dot{n}^{\kappa\mu}}\dot{n}^{\kappa\mu}\right) - \frac{\partial L'}{\partial t} = 0$$

(with implied summation) holds as an identity, the DuBois–Reymond condition (Cesari, 1983, sect. 2.2). For that reason we seek a coordinate transformation to an L'' for which $\partial L''/\partial t = 0$. We know that $\partial\rho/\partial t = 0$ for $\rho = \rho(q_e, k; p_{-1}, c)$, and we are at liberty to transform L' in accordance with (A-4). Consider, therefore,

$$n''^{\kappa\mu} = \exp\left(\int_{t_\alpha}^{t}\left(\rho - \frac{d_{\kappa\mu}}{p_{\kappa\mu}}\right)d\tau\right)n^{\kappa\mu} \qquad \kappa = 1,\ldots,K;\ \mu = 1,\ldots,M,$$

where $d_{\kappa\mu}/p_{\kappa\mu} = d'_{\kappa\mu}/p'_{\kappa\mu}$, implying

$$\begin{aligned} \dot{n}^{\kappa\mu} &= \exp\left(-\int_{t_\alpha}^{t}\left(\rho - \frac{d_{\kappa\mu}}{p_{\kappa\mu}}\right)d\tau\right)\dot{n}''^{\kappa\mu} \\ &\quad -\left(\rho - \frac{d_{\kappa\mu}}{p_{\kappa\mu}}\right)\exp\left(-\int_{t_\alpha}^{t}\left(\rho - \frac{d_{\kappa\mu}}{p_{\kappa\mu}}\right)d\tau\right)n''^{\kappa\mu} \end{aligned}$$

($\kappa = 1,\ldots,K;\ \mu = 1,\ldots,M$), and

$$p'_{\kappa\mu} = \exp\left(+\int_{t_\alpha}^{t}\left(\rho - \frac{d_{\kappa\mu}}{p_{\kappa\mu}}\right)d\tau\right)p''_{\kappa\mu},$$

since $p_{\kappa\mu}$ must transform according to the rules

$$p_{\kappa\mu} = p^{\oplus}_{\kappa\mu} = -(1/h)\,\partial L\left(t, n^{\oplus}, \dot{n}^{\oplus}\right)/\partial\dot{n}^{\kappa\mu},$$

and $d_{\kappa\mu} = (1/h)\,\partial L/\partial n^{\kappa\mu}$ transforms in the same way.

Then L' is transformed to

$$\begin{aligned} L'' &= d''_{\kappa\mu}n''^{\kappa\mu}h - p''_{\kappa\mu}\frac{dn''^{\kappa\mu}}{dt}h + \left(\rho - \frac{d_{\kappa\mu}}{p_{\kappa\mu}}\right)p''_{\kappa\mu}n''^{\kappa\mu}h \\ &= \rho\, p''_{\kappa\mu}n''^{\kappa\mu}h - p''_{\kappa\mu}\frac{dn''^{\kappa\mu}}{dt}h, \end{aligned}$$

from which, by the DuBois-Reymond condition, it follows that

$$\frac{d}{dt}\left(\rho\, p''_{\kappa\mu}n''^{\kappa\mu}h\right) = 0,$$

or equivalently

$$\begin{aligned} \frac{dY''_t}{dt} &= 0; \\ Y''_t &= \rho\, p''_{\kappa\mu}n''^{\kappa\mu}h = \rho\exp\left(-\int_{t_\alpha}^{t}\rho\, d\tau\right)p_{\kappa\mu}n^{\kappa\mu}h. \end{aligned}$$

Thus, we see that the inclusion of capital gains (the transformation from current to total return) can be viewed as a time-dependent coordinate transformation; and the present value total return on the aggregate financial asset portfolio constitutes a time-invariant quantity.

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