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An Intertemporal General Equilibrium Model of Financial Crowding Out

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I. Introduction

The aim of this paper is to construct an economic model designed to estimate the extent to which public spending crowds out private production and capital formation. Although the analysis is purely theoretical, it should be seen as a first step in building an empirical model. The approach is especially useful for policy analysis, since it simultaneously allows the consideration of disaggregated fiscal measures, such as changes in individual tax rates or transfer payments, yet at the same time incorporates macroeconomic aspects of fiscal policy, such as the rules for deficit financing and the interaction between government deficits, interest rates, and inflation. In addition, by disaggregating the private sector, a comparison can be made of the extent to which individual industries suffer (or benefit) from public sector spending policies. Although, as with all economic models applied to real situations, a certain degree of skepticism would be required to accept the results derived from simulations of this model, the estimation requirements are not significantly greater than those in a number of currently existing and accepted models. 1/ Thus, the policy conclusions resulting from the model should, at the very least, offer useful guidance to policymakers.

Before turning to a description of the model, it may be useful to review briefly certain aspects of the current literature on crowding out, so as to point out the differences of the model. The issue of crowding out has usually been examined in two different but related contexts.

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1/ See Shoven (1982) for a survey of some of these models.

In the first, the public sector purchases large quantities of goods and finances these purchases either by taxes or by borrowing. Insofar as these purchases are used for the production of public goods, they will no longer be available as inputs in the production of the private sector, whose output will therefore be forced to decline. The second such context, usually referred to as financial crowding out, concerns the government's increasing its borrowing requirements, thereby driving up the interest rate. Access to credit markets is thus made more expensive to the private sector, so that it is forced to curtail that part of its capital formation that is not self-financed. Indirect crowding out may also occur as rising interest rates may cause current consumption, and hence demand for the output of the private sector, to fall.

Financial crowding out has traditionally been analyzed within the context of macroeconomic models in which the private sector is aggregated into a single unit, private capital formation is dependent upon the interest rate, and the interest rate is, in turn, dependent upon the government's borrowing requirements and, hence, its deficit. 1/ There are severe limitations to this aggregative approach. Borrowing requirements are different across industries so one would expect the government's borrowing to have a differential impact on the private sector. The aggregation of demand also precludes any analysis of the relative impact of government fiscal policies on the welfare of different consumer groups. In addition, as the models are usually valid only for small changes, it is difficult, if not impossible, to estimate the impact of sudden, rapid increases in government borrowing. Often, governments attempt to increase simultaneously tax revenue and borrowing. Because the macroeconomic models in question do not normally separate tax revenues and government expenditure, such policies cannot be properly dealt with. 2/ Despite such limitations, macroeconomic models of this type have been used widely to give policy advice, often in circumstances in which their underlying assumptions cause them to be not strictly valid.

The question of resource crowding out is increasingly being examined within the framework of computational general equilibrium (CGE) models of taxation. Such models, originally inspired by the work of Harberger (1962, 1966) on tax incidence, have been developed in Shoven and Whalley

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1/ Among such models are those of Blinder and Solow (1973, 1974), Brunner and Meltzer (1972), Buiter (1977), Christ (1968), Cohen and McMenamin (1978), Friedman (1978), Gramlich (1971), Infante and Stein (1976), Meyer (1975), Modigliani and Ando (1976), Spencer and Yohe (1970), and Tobin and Buiter (1976).

2/ See Tanzi (1978) and Aghevli and Khan (1978) for models which do distinguish between taxes and expenditures.

(1972, 1973), Shoven (1976), Fullerton (1982 a, 1982 b), Fullerton and others (1981), Miller and Spencer (1977), Piggott and Whalley (1983), and Whalley (1975, 1977, 1982), among others, to examine incidence and welfare implications of changes in tax regimes. The advantages of these models, as compared with the macroeconomic ones, have been discussed at length in Shoven (1982); among them is the ability to deal with large changes in government policies, with disaggregated taxes, and with the analysis of the welfare implications of taxation through the examination of individual consumer categories. There are, however, a number of disadvantages. These models have been almost exclusively "real," so that the public sector is constrained to have a balanced budget, owing to the absence of financial assets that could be used to finance a deficit. Because there is no money, and hence no price level or interest rate, it is impossible to analyze financial crowding out. From the point of view of the government policymakers, the advice given by such models must be quite suspect. If their results are to be believed, it is then necessary to believe that government deficits have no real impact; if they did, the balanced budget results produced by the models would be meaningless.

Research in which certain types of CGE models are expanded to include financial assets has recently been carried out by several authors. Clements (1980) allows for domestic credit expansion in a model of the United States, although it is exogenous with respect to public sector expenditure and revenues. Feltenstein (1980), in a model of Argentina, permits the existence of domestic and foreign financial assets, whose endogeneity of supply is dependent upon the balance of payments, while Feltenstein (1984) has an endogenous government deficit and corresponding financing through the issuance of money and bonds. Another approach has been that of Slemrod (1981), who constructs a CGE model incorporating portfolio choice by consumers. For the policymaker the major flaw in these models is that they do not permit both endogenous public deficits and private investment, and therefore cannot adequately cope with the issue of crowding out. We will present here a model of a closed economy, intended to address this flaw, which has a computational general equilibrium structure, but which also has considerable macroeconomic content.

An earlier version of this model was presented in Feltenstein (DM/83/1). That paper contained, however, several unsatisfactory elements. Because there were no constraints on the spending behavior of the central government, it was not possible to ensure that the public would be willing to hold the real quantity of debt needed to finance the government's deficit. In addition, because there were also no constraints on the government's issuance of money, it was possible to create a hyperinflationary situation in which the government drives the rate of inflation to infinity by trying to force a larger real quantity

of money onto the public than the public is willing to hold. A further shortcoming of this model was its treatment of private investment. The value of private investment was derived as the difference between private savings and the value of the government's debt issuance, so that the Keynesian identity was satisfied at any instant in time. There was, thus, no behavioral representation of investment connecting it to the interest rate and the anticipated rate of return, and crowding out was given automatically by the method of deriving private investment. The model constructed here is dynamic; it has two periods with the notion of a past (before period 1) and a future (after period 2). Both consumers and firms have perfect foresight for the two periods, so that the prices, tax liabilities, and transfers received from the government in period 2 are correctly anticipated in period 1. 1/ In the future (after period 2) consumers become perfectly myopic, expecting the same structure of prices, taxes, and government transfers to prevail then as in period 2. 2/

Firms in the private sector are constrained to cover current expenditures by current revenue, while capital formation is financed by the sale of bonds. The government, on the other hand, sets its program of expenditure in real terms and is not required to cover costs from tax revenues, and when it incurs a deficit, the government issues a combination of money and bonds to cover its loss. The government is sensitive, however, to the impact that its deficits may have upon interest and inflation rates. Accordingly, it will gradually cut its spending as real interest and inflation rates rise above predetermined targets. Consumers are required to hold money to cover transaction costs, and they purchase bonds in order to save for the future. Perfect foresight precludes the possibility of risk, so that private and government bonds are viewed by the consumer as being identical. The equilibrium condition on privately issued debt is that new capital produced in period 1, which comes on line in period 2, must yield a return in period 2 equal to the obligations on the bonds that financed it. The government, on the other hand, must add the debt obligations incurred in period 1 and coming due in period 2 to its current expenditures in that period.

1/ The model may thus be interpreted as generating a rational expectations equilibrium, in which consumers have no incentive to revise their expectations of the future, having correctly anticipated period 2. The minimum length of time needed to introduce a dynamic framework is two periods, but there would be no difficulty in extending the model to several periods.

2/ This "closure" rule is made for purely technical reasons. We must allow for some future after the final period in order to avoid the requirement that in that final period there be a balanced government budget and no private investment, as consumers would, in the absence of a future, refuse to hold debt.

Consumers, who are disaggregated, maximize intertemporal utility functions and derive a demand for bonds as a method of savings. Simultaneously, the government, when it runs a deficit, sells bonds at a discount. The amount of discount becomes greater the larger the deficit is. As the corresponding interest rate becomes higher, consumers satisfy the Fisherian relation and shift their consumption to the future, releasing resources to the government. These resources, in particular, savings, are increasingly unavailable to the private sector, which is also constrained by the fact that the debt obligations it is incurring for period 2 are rising relative to the anticipated rate of return on its investment in period 1. The private sector thus suffers from both resource and financial crowding out. 1/

The model includes profit, income, and sales taxes and allows for direct transfer payments by the government to consumers. The price level is endogenous, so that the inflationary impact of various government policies may be analyzed. There is also an investment function, with the level of investment being driven by the interest rate. The model would therefore lend itself to empirical implementation, as such functions, along with that representing the production technology, are commonly estimated.

Section II will present a formal description of the model, while Section III will demonstrate the existence of an equilibrium. Section IV will be a conclusion, indicating certain directions for empirical implementation of the model and for future research.

II. The Model

1. Production

The structure of production is Leontief in intermediate and final production, while value added is produced by smooth production functions. 2/ Because the model incorporates perfect foresight in both production and consumption, production may be represented by a block-diagonal matrix, whose components refer to goods that are different in their dating. 3/ If goods $i = 1, \ldots, N$ refer to goods produced in

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1/ It should be emphasized, however, that the model does not yield a mechanical one-to-one correspondence between public deficits and crowding out, because the rising interest rate will not only have the above-mentioned effects but also will increase the overall level of savings.

2/ This formulation is used because of the eventual goal of an empirical application and has been described in greater detail in, for example, Fullerton and others (1981) and Feltenstein (1980).

3/ See Debreu (1959) for a discussion of the use of dated commodities.

period 1 and goods N + 1, ..., 2N refer to goods produced in period 2, then the structure of the production matrix for intermediate and final goods is

$$
(1) \quad \begin{array}{ccccc}
a_{11}, & \dots, & a_{1N}, & 0, \dots, & \dots, & 0 \\
\cdot & & & & & \cdot \\
\cdot & & & & & \cdot \\
a_{N1}, & \dots, & a_{NN}, & 0, \dots, & \dots, & 0 \\
0, & \dots, & 0, & a_{N+1,N+1}, & \dots, & a_{N+1,2N} \\
\cdot & & \cdot & \cdot & & \cdot \\
\cdot & & \cdot & \cdot & & \cdot \\
\cdot & & \cdot & \cdot & & \cdot \\
0, & \dots, & 0, & a_{2N,N+1}, & \dots, & a_{2N,2N}
\end{array}
$$

The upper block of the matrix refers to first period production, and the lower block refers to second period production. If there is no technological change between the two periods, then the coefficients in the two blocks would be identical. Corresponding to each activity, there is a continuous function $f_j(K_i, L_i)$, which produces value added for the j^{th} activity using capital and labor from the corresponding stocks that exist in period i. In order to be specific, assume that the value-added functions are Cobb-Douglas, hence of the form

$$(2) \quad f_j(K_i, L_i) = K_i^{\alpha_j} L_i^{(1-\alpha_j)}$$

In addition, there are investment activities, $H_i(K_i, L_i)$, which operate in period i, using inputs of capital and labor existing in that period, and which produce capital goods for period i+1. 1/ The investment is considered to be part of the private sector, and since the capital that

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1/ The investment function could also require intermediate and final goods as inputs, but for simplicity of exposition, it will require only capital and labor as inputs.

is produced only becomes available in the next period, the investment firm must pay for the input costs of its production in the current period but will receive the revenue from that capital in the next period only. 1/ In order to simplify the demonstration of the existence of an equilibrium, it is supposed that the investment functions exhibit decreasing returns to scale, and, again to be specific, that they are of the form 2/

$$(3) \quad H_i(K_i, L_i) = K_i^{a_i} L_i^{b_i} \;; \qquad a_i + b_i < 1 \quad a_i, b_i > 0$$

Capital in period 2 is then given by the depreciated initial capital stock plus whatever new capital has been produced in period 1. If $\bar{K}_0$ is the initial stock of capital at the beginning of period 1, δ the rate of depreciation, and $\bar{L}_0$ is the initial stock of labor, then

$$(4) \quad K_2 = (1-\delta)\bar{K}_0 + H_1(\bar{K}_0, \bar{L}_0)$$

$$K_f = (1-\delta)K_2 + H_2(K_2, \bar{L}_0)$$

where K_2 is the stock of capital at the beginning of period 2, and K_f is the capital stock existing in the future (after period 2). 3/

1/ It would be possible to have investment activities distinguished by firms if we also had firm-specific capital, as in Fullerton (1982), and Dervis, DeMelo, and Robinson (1982), but to do so would not qualitatively change the nature of the model.

2/ Decreasing returns to scale will allow the derivation of a single-valued investment response. If desired, we could choose the parameters such that $1-a_i-b_i = \varepsilon_i$ with ε_i arbitrarily small. Any decreasing returns to scale investment function would be equally acceptable.

3/ To avoid introducing a differential age structure into the model, zero growth in the population is assumed. There is, thus, only a single generation alive at one time.

The government also produces public goods, for which it receives no revenue, through a smooth production function that uses capital and labor of the current period as inputs. 1/ Let $Q_i(K_i, L_i)$ denote this function in period i and, for simplicity, also assume the function to be Cobb-Douglas, hence of the form

$$(5) \quad Q_i(K_i, L_i) = K_i^{\beta_i} L_i^{(1-\beta_i)}$$

The government is assumed to decide, at the beginning of period i, on the level of output of public goods, in real terms, to be produced during the period. The government then solves the equation

$$(6) \quad Q_i = K_i^{\beta_i} L_i^{(1-\beta_i)}$$

where Q_i is the real quantity of public goods to be produced in period i, in such a way as to minimize the cost of production. The financing of the cost of this production will be discussed in Section II.3, but it should be noted here that the government issues money and bonds, which are also sold by private investment activity.

2. Consumption

The consumers in the model are viewed as living for the entire period of the model, namely, for the two periods being solved and also for the third, or future period. Since they may have initial endowments of goods other than labor, it is implicitly supposed that they were alive before period 1, so that their holdings of capital and financial assets may be carried over into period 1. In periods 1 and 2 the consumers have perfect foresight, that is, they perfectly anticipate all prices

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1/ Rather than having the government operate its own production function, it would also be possible to have the government buy directly from the private sector. Introducing a government production function allows, however, the direct representation of changing public policy toward the relative importance of hiring capital or labor. If, for example, the government wished to increase employment, it could, in the model, change the weights given to capital and labor in its production function.

of period 2 while they are still in period 1. 1/ They also correctly anticipate their tax obligation (or transfer payments received) in period 2. In the future (after period 2) the consumers become perfectly myopic, by which we mean that they anticipate the same relative prices and tax obligations (or transfer payments) will hold in the future as held in period 2. These obligations and prices will simply be scaled up by whatever the anticipated rate of inflation is. An interpretation of this type of expectation is that, after having been proven to be correct in period 2, the consumers believe that the economy is on a steady-state growth path.

The individual consumer maximizes a utility function, U_i, which has as arguments the levels of consumption in each of the two periods. 2/ Thus,

(7) $U(x) = U(x_1, \ldots, x_N, x_{N+1}, \ldots, x_{2N}, x_{L1}, x_{L2})$

where x_i: $i < N$ refers to the i^{th} consumption good in period 1, x_i: $i > N$, refers to the i^{th} consumption good in period 2, and L_i refers to consumption of leisure in period i. In order to be specific, the utility function is assumed to be of the form

(8) $U = x_1^{d_1} x_2^{d_2}, \ldots, x_{2N}^{d_{2N}} x_{L1}^{d_{L1}} x_{L2}^{d_{L2}}$

1/ A rational expectations equilibrium is being defined in which consumers' expectations of period 2 are perfectly fulfilled, so that they have no incentive to revise these expectations in the future. If the model contained more than two periods, it would be quite possible that information available for the time period after period 2 might be used to determine the consumers' choices in periods 1 and 2.

2/ There are $K > 0$ consumers in the model; however, in order to avoid unreadable subscripts the consumer demand parameters will not be indexed. It should be noted that these parameters, along with initial allocations, are not uniform across consumers. One might also wish to include public goods in the consumers' utility function. Since the concern in this paper is in the financing of investment and the government, rather than in the choice of public goods, the consumer's utility function has not been included.

where d_i: $i = 1, \ldots, 2N+2$ are the expenditure shares given to consumption goods, including leisure. Suppose that these expenditure shares reflect the consumer's rate of time preference, z, so that

$$(9) \quad \sum_{i=1}^{2N} d_i + \sum_{i=1}^{2} d_{Li} = 1 : d_i > 1$$

$$\sum_{i=1}^{N} d_i + d_{L1} = z \left\{ \sum_{i=N+1}^{2N} d_i + d_{L2} \right\} : z > 1$$

and, in addition, z is uniform across all consumers. None of these restrictions are essential to the working of the model, but they correspond to the normal macroeconomic interpretation of time preference. Hence, leisure enters the utility function, but money, bonds, and capital do not. It should be noted that our proof of the existence of equilibrium does not depend on this form of the utility function; any continuous utility function would be valid. This particular form permits an analytic solution to the demand function.

The consumer maximizes his utility function, subject to a set of intertemporal budget constraints, as it is assumed that capital markets are imperfect in that consumers cannot borrow against future income. The consumer must, therefore, cover his current expenditure plus savings from current income. 1/ He has an initial allocation of money and bonds, $\bar{M}_0$ and $\bar{B}_0$, at the beginning of period 1, and, if he is a shareholder in the capital goods-producing form, he will also hold capital $\bar{K}_0$. 2/ Let p_{Ki}, p_{Li}, p_{Mi}, p_{Bi} represent the prices of capital, labor, money,

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1/ Another approach in, for example, Grandmont (1977) and Grandmont and Laroque (1975), is to have consumers borrow from the central bank against future income but to have no borrowing by the central bank. A number of technical problems are involved with allowing borrowing to go in both directions, essentially equivalent to the requirement of irreversibility of production.

2/ It will be assumed that initial holdings of bonds, $\bar{B}_0$, are entirely composed of government debt. As shall be seen in Section II.3, initial private debt would be inconsistent with the specified intertemporal investment decision.

and bonds, respectively, in period i, and let TR_i represent whatever transfer payments the government pays to consumers during period i, while γ_i, represents this particular consumer's share in those transfers. 1/ Since the model has many consumers, ideally, all allocations, demand, and share parameters should be indexed to refer to the individual consumer. Thus, for example $\bar{M}_0^j$ would refer to the j^{th} consumer's initial holdings of money, while γ_i^j would refer to the j^{th} consumer's share in government transfers in period i. As shall be seen, however, such a convention would lead to a very unwieldy system of subscripts and superscripts. Therefore, when reference is made to the individual consumer's maximization problem, superscripts will not be used. When aggregate market demands (see equation 38) are derived, however, superscripts will be introduced.

Bonds are considered to be long term, so that a consumer owning a bond receives its par value as an interest payment in each period that he owns the bond. Since this payment is made in units of money, his income from the bond in period i is p_{Mi}. He also has the possibility of selling the bond at market prices p_{Bi}. The consumer's income, $I_1(p_1, p_2)$, in period 1, is then given by

$$(10)\quad I_1(p_1,p_2) \equiv p_{M1}\bar{M}_0 + p_{M1}\bar{B}_0 + p_{B1}\bar{B}_0 + p_{K1}\bar{K}_0 + p_{L1}\bar{L}_0 + \gamma_1 TR_1$$

In addition, the consumer has a second period budget constraint. If he has purchased a quantity, x_{B1}, of bonds in period 1, he then receives the coupon value of those bonds in terms of units of money in period 2, this being equal to $p_{M2}x_{B1}$, if it is assumed that the coupon payment is 1. 2/ The consumer's income in period 2, $I_2(P_1,P_2)$, then becomes

$$(11)\quad I_2(p_1,p_2) = p_{K2}(1-\delta)\bar{K}_0 + p_{L2}\bar{L}_0 + p_{M2}x_{M1} + p_{M2}x_{B1} + p_{B2}x_{B1} + \gamma_2 TR_2$$

1/ The share could thus change from period to period.

2/ The rate of inflation is defined as $p_{M1}/p_{M2}-1$, the percentage change in the price of money. Thus, an indexed bond would yield a coupon payment of p_{M1}, while a nonindexed bond, as we are considering, would pay p_{M2} in period 2.

where x_{M1} is the quantity of money that the consumer holds in period 1.

By supposing also that the bonds purchased in period 1 are long term, in the sense that they are not redeemable during the time span of the model, but continue to pay a uniform coupon, an important assumption is thus being made, namely, that there is no secondary market for capital, so that a consumer who holds capital cannot sell his capital to either other consumers or to enterprises. 1/ If we had a model with multiperiod overlapping generations, then the possibility of one generation selling its capital to a new generation would have to be accounted for. Such a sale would take place when the rental stream of future earnings on the capital is discounted by the new generation over their life span by the future interest rate and is found to be at least equal to the sales price of capital. In a single generation model, however, the same discounting would be carried out by all consumers, so that no sales of capital would take place. *The major implication* of this assumption is that capital gains will be realized in the model only through the sale of bonds. *A capital gains tax has not been* introduced, although doing so would pose no technical problem.

Although a third period is not explicitly solved, the model does have the notion of a future that is essentially the same as period 2. Thus, the consumer will expect that the same relative prices will prevail in this future as existed in period 2, but that they will increase by whatever the expected rate of inflation is. The assumption then is that the consumer wishes to purchase the same bundle of consumption in the future as he purchased in period 2, subject to his rate of time preference. 2/ His expected future income, $I^E(p^E)$ is given by

$$(12)\quad I^E(p^E) = p_k^E(1-\delta)^2\bar{K}_0 + p_L^E\bar{L}_0 + p_M^E x_{M2} + p_M^E x_{B2} + p_B^E x_{B2} + \gamma_2 TR^E$$

where the subscript E denotes the expected value of the corresponding variable. In the third, or future period, the consumer will continue to receive the coupon payments from the bonds he held in period 2.

1/ The interpretation of the price of capital in period i, p_{Ki}, is that it is a rental rather than a sales, or cost of production, price. It should also be noted that if this model extended backwards to the point where the existing capital stock began to be created, then ownership of capital would be fully represented by ownership of bonds.

2/ As mentioned earlier, this assumption is being made simply to close the model.

The consumer is assumed to be myopic in the future, so that he expects no change in relative prices of goods and bonds. He does, however, anticipate some rate of inflation, reflected in the depreciated purchasing power of money in the future. Therefore, in particular,

$$(13)\ p_K^E = (1+\pi^E)p_{K2}^E,\ p_L^E = (1+\pi^E)p_{L2},\ p_M^E = p_{M2},\ p_B^E = (1+\pi^E)p_{B2},$$

$$TR^E = (1+\pi^E)TR_2$$

where π^E is the expected rate of inflation. Although the method of derivation of π^E is not relevant to this study, one might continue to assume myopia, so that π^E is equal to the actual rate of inflation between periods 1 and 2, inflation being defined as the rate of change in the price of money. 1/ Hence,

$$(14)\ \pi^E \equiv p_{M1}/p_{M2} - 1$$

The consumer, in solving his utility maximization problem, has three simultaneous budget constraints, one for each of the two perfectly anticipated periods, and one for the future period. Suppose that the consumer faces ad valorem taxes on his purchases of consumption goods, and let

$$(15)\ t_1 \equiv (\tau_1, \ldots, \tau_N) \qquad : 0 < \tau_i$$

$$t_2 \equiv (\tau_{N+1}, \ldots, \tau_{2N}) \quad : 0 < \tau_i$$

where t_i represents the vector of tax rates levied on the N intermediate and final goods produced in period i. Let

$$(16)\ \tilde{p}_1 \equiv (p_1, \ldots, p_N),\ \tilde{p}_2 \equiv (p_{N+1}, \ldots, p_{2N})$$

1/ It would be more correct to define inflation in terms of a consumer price index, rather than a one-commodity basket. Doing so would, however, require the introduction of index weights.

denote the prices of the intermediate and final goods in each of the two periods. The value of the consumer's expenditure on all goods, including leisure and bonds, in period i is then given by

$$(17)\ (1+t_i)\tilde{p}_i \cdot x_i + p_{Li}x_{Li} + p_{Bi}x_{Bi}$$

where x_i, x_{Li}, x_{Bi}, represent his consumption of goods, leisure, and bonds in period i, respectively. The consumer, in addition, requires a certain quantity of money to cover transaction costs. In Feltenstein (1983) this transaction demand is presented as a constant fraction of the value of consumption, representing, in other words, a constant velocity of money.

Here, a somewhat more realistic version of the demand for money is presented, in which demand for nominal cash balances depends not only on the value of current consumption but also on the nominal interest rate. Suppose a simple quantity theory of money which, in the system, would be formulated as

$$(18)\ p_{Mi}x_{Mi} = \frac{1}{v_i}(1+t_i)\tilde{p}_i \cdot x_i \quad : \quad v > 1$$

where v_i is the velocity of money in period i. Thus, the nominal value of money demanded in period i is a function of the value of consumption of intermediate and final goods in that period. Leisure is not included as a determinant in the demand for money, since income taxes, as will be discussed shortly, are withheld at the source, that is, the firm.

Suppose now that v_i is not constant, but is a function of the nominal interest rate. The nominal interest rate, r_1, or the percentage return on a bond, in period 1 is given by

$$(19)\ r_1 = (p_{M2} - p_{B1})/p_{B1}$$

while in period 2 it is given by

$$(20)\ r_2 = (p_M^E - p_{B2})/p_{B2}$$

since the bond pays a unitary coupon (in terms of units of money) in the periods after which it is purchased. 1/ Suppose also that

$$(21)\quad v_i = \frac{1}{a} e^{br_i} \quad : a, b > 0$$

so that the velocity of money is directly related to the nominal interest rate. Hence, by substitution,

$$(22a)\quad p_{Mi}x_{Mi} = ae^{-br_i}(1+t_i)\tilde{p}_i \cdot x_i \quad : \quad ae^{-br_i} < 1$$

$$= (1+t_i)\tilde{p}_i x_i \quad : \quad ae^{-br_i} > 1$$

In the next section, it will be demonstrated that the government's issuance of money in period i, $\tilde{y}_{Mi}$, is bounded for any set of prices, i.e., $\tilde{y}_{Mi} < \bar{y}_{Mi}$ for some $\bar{y}_{Mi} < \infty$. In order to demonstrate the boundedness of the intertemporal excess demand functions, we will assume that this upper bound is known to consumers and that the individual consumer will not demand more money than the upper bound of the supply. Hence,

$$(22b)\quad x_{M1} = \frac{ae^{-br_1}(1+t_1)\tilde{p}_1 \cdot x_1}{p_{M1}} \quad : x_{M1} < \bar{M}_0 + \bar{y}_{M1}$$

$$= \bar{M}_0 + \bar{y}_{M1} \quad \text{otherwise}$$

$$x_{M2} = \frac{ae^{-br_2}(1+t_2)\tilde{p}_2 \cdot x_2}{p_{M2}} \quad : x_{M2} < K(\bar{M}_0 + \bar{y}_{M1}) + \bar{y}_{M2}$$

$$= K(\bar{M}_0 + \bar{y}_{M1}) + \bar{y}_{M2} \quad \text{otherwise. } \underline{2/}$$

1/ The bond is thus a console whose nominal interest payment is given by its par value, which is fixed in terms of units of money. The real value of the interest payments will, of course, decline with inflation.

2/ The next section will show why the second period constraint takes this form. The individual consumer need not know the actual bounds $\bar{y}_{Mi}$, but only that there are such bounds and must bound his demand for money accordingly. Clearly, this restriction is not relevant in any realistic situation.

Things may be cut in a somewhat more familiar form by taking logarithms,

$$(23)\quad \ln(p_{Mi}x_{Mi}) = \ln a - br_i + \ln(1+t_i)\tilde{p}_i \cdot x_i$$

The total value of the consumer's period 1 and period 2 consumption must be equal to or less than the corresponding income, hence

$$(24)\quad (1+t_i)\tilde{p}_i \cdot x_i + p_{Li}x_{Li} + p_{Bi}x_{Bi} + p_{Mi}x_{Mi} \leq I_i(p_i)$$

where $p_{Mi}x_{Mi}$ is given by equation (22) and $I_i(p_i)$ is given by equations (10) and (11). For the third, or future, period the behavioral assumption is made that the consumer wishes to be able to purchase in the future the same bundle of consumption as he bought in period 2, discounted by his rate of time preference. 1/ Accordingly,

$$(25)\quad \frac{[(1+t_2)\tilde{p}_2 \cdot x_2 + p_{L2}x_{L2} + p_{M2}x_{M2}](1+\pi^E)}{z} = I^E(p^E)$$

where $I^E(p^E)$ is defined in equation (12). Bonds are not included in equation (25), because to do so would presuppose an anticipated period beyond the third, or future, period. The entire value of consumption, including the value of money holdings, in the future is increased by the rate of inflation, as we assume that the same relation between the value of consumption goods and money holdings exists in the future as in period 2.

Consolidating equations (10), (11), (12), (22), and (25) results in the following maximization problem for the consumer.

$$(26)\quad \max\ x_1^{d_1}\ x_2^{d_2}\ \cdots\ x_{2N}^{d_{2N}}\ x_{L1}^{d_{L1}}\ x_{L2}^{d_{L2}}$$

1/ This assumption follows from the fact that his myopic expectations after period 2 cause him to anticipate the relative prices of period 2 to prevail in the future.

such that

$$(26a)\quad (1+t_1)\tilde{p}_1x_1 + p_{L1}x_{L1} + p_{B1}x_{B1} + ae^{-br_1}(1+t_1)\tilde{p}_1x_1$$

$$< p_{M1}\bar{M}_0 + p_{K1}\bar{K}_0 + p_{M1}\bar{B}_0 + p_{B1}\bar{B}_0 + p_{L1}\bar{L}_0 + \gamma_1 TR_1$$

$$(26b)\quad (1+t_2)\tilde{p}_2x_2 + p_{L2}x_{L2} + p_{B2}x_{B2} + ae^{-br_2}(1+t_2)\tilde{p}_2x_2$$

$$< p_{K2}(1-\delta)\bar{K}_0 + p_{L2}\bar{L}_0 + p_{M2}x_{M1} + p_{M2}x_{B1} + p_{B2}x_{B1} + \gamma_2 TR_2$$

$$(26c)\quad \frac{(1+t_2)\tilde{p}_2x_2 + p_{L2}x_{L2} + p_{M2}x_{M2}}{z} = p_{x2}(1-\delta)^2\bar{K}_0 + p_{L2}\bar{L}_0 + p_{M2}x_{M2}$$

$$+ p_{B2}x_{B2} + \gamma_2 TR_2$$

Along with the constraint on the individual consumer's demand for money described in equation (22b), a constraint is also imposed on the individual's demand for bonds. It will be shown that there is an upper bound on the supply of bonds, $\bar{y}_{Bi}$, in period i, where $\bar{y}_{Bi}$ incorporates the debt issued by both the public and the private sectors. The assumption is therefore made that this upper bound is known to individual consumers so that

$$(26d)\quad x_{Bi} < \bar{y}_{Bi}$$

where x_{Bi} are the holdings of bonds by a particular consumer in period i.

Again, it should be noted that the consumer need not know the precise upper bounds to the supply of bonds, only that there are such bounds and to behave accordingly.

The maximization problem in equation (26) may then be put in the form of a Lagrangian as

$$(27)\quad L = x_1^{d_1} \cdots x_N^{d_N}\, x_{L1}^{d_{L1}}\, x_{N+1}^{d_{N+1}} \cdots x_{2N}^{d_{2N}}\, x_{L2}^{d_{L2}}$$

$$+\left\{\lambda_1\, p_{M1}\bar{M}_0 + p_{L1}\bar{L}_0 + p_{K1}\bar{K}_0 + p_{M1}\bar{B}_0 + \gamma_1 TR_1 - \left[1 + ae^{-br_1}\right](1+t_1)\tilde{p}_1 x_1 - p_{L1}x_{L1} - p_{B1}x_{B1}\right\}$$

$$+ \lambda_2\left\{(1-\delta)p_{K2}\bar{K}_0 + p_L\bar{L}_0 + \frac{p_{M2}}{p_{M1}}\, ae^{-br_1}(1+t_1)\tilde{p}_1 x_1 + (p_{M2}+p_{B2})x_{B1} + \gamma_2 TR_2\right.$$

$$\left. - \left[1 + ae^{-br_2}\right](1+t_2)\tilde{p}_2 x_2 - p_{L2}x_{L2} - p_{B2}x_{B2}\right\}$$

$$+ \lambda_3\left\{zp_{K2}(1-\delta)^2 K_0 + zp_{L2}\bar{L}_0 + z\left(\frac{p_{M2}}{1+\pi^E} + p_{M2}\right)x_{B2} + zr_2 TR_2\right.$$

$$\left. - \left[1 + (1+\pi^E - z)ae^{-br_2}\right](1+t_2)p_2x_2 - p_{L2}x_{L2}\right\}$$

The constraints of equations (22b) and (26d) are not incorporated here so as to simplify the final expression we derive. Their inclusion does not, however, pose any problem. Solving,

$$(28)\quad \frac{\partial L}{\partial x_j} = d_j x_1^{d_1} x_2^{d_2} \cdots x_j^{d_j - 1} \cdots x_{L2}^{d_{L2}} - \lambda_1\left\{1 + ae^{-br_1}\right\}(1+t_j)p_j$$

$$+ \lambda_2\left\{\frac{p_{M2}}{p_{M1}}\, ae^{-br_1}\right\}(1+t_j)p_j = 0 \quad : \quad 1 < j < N$$

$$\frac{\partial L}{\partial x_{L1}} = d_{L1} x_1^{d_1} \cdots x_N^{d_N}\, x_{L1}^{d_{L1}-1} \cdots x_{L2}^{d_{L2}} - \lambda_1 p_{L1} = 0$$

$$\frac{\partial L}{\partial x_j} = d_j x_1^{d_1} \cdots x_j^{d_j - 1} \cdots x_{L2}^{d_{L2}} - \lambda_2 \left[1 + ae^{-br_2}\right](1+t_j)p_j$$

$$- \lambda_3 \left[1 + (1+\pi^E - z)ae^{-br_2}\right](1+t_j)p_j = 0 :$$

$$N+1 < j < 2N$$

$$\frac{\partial L}{\partial x_{L2}} = d_{L2} x_1^{d_1} \cdots x_{2N}^{d_{2N}} x_{L2}^{d_{L2}-1} - \lambda_2 p_{L2} - \lambda_3 p_{L2} = 0$$

$$\frac{\partial L}{\partial x_{B1}} = - \lambda p_{B1} + \lambda_2 (p_{M2} + p_{B1}) = 0$$

$$\frac{\partial L}{\partial x_{B2}} = - \lambda_2 p_{B2} + \lambda_3 z \left(\frac{p_{M2}}{1+\pi^E} + p_{M2}\right) = 0$$

Thus, in particular:

$$(29) \quad x_j = \frac{d_j}{d_1} \frac{(1+t_1)}{(1+t_j)} \frac{p_1}{p_j} x_1 \quad : \quad 1 < j < N$$

$$x_{L1} = \frac{d_{L1}}{d_1} \frac{U_1 (1+t_1) p_1 x_1}{p_{L1}} \quad : \quad U_1 \equiv 1 + \frac{(p_{M2}+p_{B1})p_{M1} - p_{M1}p_{M2}e^{-br_1}}{(p_{M2}+p_{B1})p_{M1}}$$

$$x_j = \frac{d_j}{d_1} \frac{U_1 (1+t_1) p_1 x_1}{U_2 (1+t_j) p_j} \quad : \quad N+1 < j < 2N$$

$$U_2 \equiv \frac{p_{B1}\left(1+ae^{-br_2}\right)}{p_{M2}+p_{B1}} + \frac{p_{B2}p_{B1}\left(1 + (1+\pi^E - z)ae^{-br_2}\right)}{z\left(p_{M2}/(1+\pi^E) + p_{M2}\right)(p_{M2}+p_{B1})}$$

$$x_{L2} = \frac{d_{L2}}{d_1}\frac{U_1(1+t_1)p_1x_1}{U_3 p_{L2}} \quad : \quad U_3 \equiv \frac{p_{B1}}{p_{M2}+p_{B1}} + \frac{p_{B2}p_{B1}}{z\left(p_{M2}/(1+\pi^E)+p_{M2}\right)(p_{M2}+p_B)}$$

Making the following definitions:

$$(30) \quad K_1 = \left(1 + e^{-br_1}\right)\left(\sum_{i=1}^{N} d_i\right) + d_{L1}U_1$$

$$K_2 = \frac{\left(1 + e^{-br_2}\right)\left(\sum_{N+1}^{2N} d_i\right)}{U_2} + \frac{d_{L2}}{U_3}$$

$$I_1 \equiv p_{M1}\bar{M}_0 + p_{L1}\bar{L}_0 + p_{K1}\bar{K}_0 + p_{M1}\bar{B}_0 + r_1 TR_1$$

$$I_2 \equiv (1-\delta)p_{K2}\bar{K}_0 + p_{L2}\bar{L}_0 + r_2 TR_2$$

$$J_1 \equiv \frac{p_{M2}}{p_{M1}} ae^{-br_1}\left(\sum_{i=1}^{N} d_i\right) - \left(\frac{p_{M2}+p_{B2}}{p_{BI}}\right)K_1 - U_1K_2$$

$$J_2 \equiv 1 + (1+\pi^E - z)ae^{-br_2}$$

$$J_3 \equiv \frac{U_1(1+t_1)}{d_1}\left\{\frac{J_2\left(\sum_{N+1}^{2N} d_i\right)}{U_2} + \frac{d_{L2}}{U_3}\right\}$$

$$J_4 \equiv zp_{K2}(1-\delta)^2K_0 + zp_{L2}\bar{L}_0 + z\left(\frac{p_{M2}}{1+\pi^E} + p_{M2}\right)\left[I_2 + \left(\frac{p_{M2}+p_{B2}}{p_{B1}}\right)I_1\right]$$

$$J_5 \equiv \frac{z\left(\frac{p_{M2}}{1+\pi^E} + p_{M2}\right) J_1(1+t_1)}{p_{B2}}$$

By solving for the remaining conditions on the Lagrangian;

$$\frac{\partial L}{\partial \lambda_1} = \frac{\partial L}{\partial \lambda_2} = \frac{\partial L}{\partial \lambda_3} = 0$$

It may be shown that the system is solved as

$$x_1 = \frac{J_4}{(J_3 - J_5)p_1}$$

and that the necessary nonnegativity conditions for x_1 are satisfied.

The demands for goods $j = 1, \ldots, 2N, L1, L2$ may then be derived from equation (29) as

$$(31) \qquad x_j = \frac{d_j}{d_1} \frac{(1+t_1)}{(1+t_j)} \frac{J_4}{(J_3-J_5)p_j} \quad : \quad 1 < j < N$$

$$x_{L1} = \frac{d_{L1}}{d_1} \frac{U_1(1+t_1)}{p_{L1}} \frac{J_4}{(J_3-J_5)}$$

$$x_j = \frac{d_j}{d_1} \frac{U_1(1+t_1)}{U_2(1+t_j)} \frac{J_4}{(J_3-J_5)p_j} \quad : \quad N+1 < j < 2N$$

$$x_{L2} = \frac{d_{L2}}{d_1} \frac{U_1(1+t_1)}{U_3 \; p_{L2}} \frac{J_4}{(J_3-J_5)}$$

The demand for money in period i, x_{Mi}, is then derived from equation (22a,b) as

$$(32) \qquad x_{M1} = ae^{-br_1} \frac{(1+t_1) \; J_4 \left(\sum_{j=1}^{N} d_j\right)}{d_1 \; (J_3-J_5)} \quad : \quad ae^{-br_1} < 1$$

$$= \frac{(1+t_1) \; J_4 \left(\sum_{j=1}^{N} d_j\right)}{d_1 (J_3-J_5)} \quad : \quad ae^{-br_1} > 1$$

$$x_{M2} = ae^{-br_2 \frac{(1+t_1)\, U_1\, J_4\left(\sum_{j=N+1}^{2N} d_j\right)}{d_1 U_2 (J_3 - J_5)}}$$

Finally, the demand for bonds in period i, x_{Bi}, may be immediately derived with the budget constraints expressed in equations (26a,b) and (26d).

Having calculated the individual consumer's demand for all goods plus financial assets, it is appropriate to turn to the derivation of aggregate supply, and, accordingly, excess demand functions.

3. Financing the central government and the formation of capital

In the model there are two production activities not required to cover current costs, the production of public goods by the central government and the production of new capital by the investment activity. Consider the case of the central government first. In order to calculate the central government's financing requirements, and hence its emission of money and bonds, its deficit (or surplus) must first be derived in each period. This deficit depends, of course, upon the tax revenues that the government collects, which, in turn, depend upon the level of supply. As before, let

$$p \equiv (\tilde{p}_1, \tilde{p}_2) = (p_{K1}, p_{L1}, p_{M1}, p_{B1}, p_{K2}, p_{L2}, p_{M2}, p_{B2})$$

be an arbitrary set of intertemporal prices for capital, labor, money, and bonds. Using the form of the individual industry's value-added functions, as given in equation (2), cost-minimizing levels of use of capital for the j^{th} sector in period i are obtained:

$$(33) \quad K_j = \left\{(1+t_{Ki}) \frac{(1-\alpha_j)}{\alpha_j} \frac{p_{K1}}{p_{L1}}\right\}^{(\alpha_j - 1)} VA_j: \quad \begin{array}{l} i = 1 \text{ if } j < N \\ i = 2 \text{ if } j > N \end{array}$$

where t_{Ki} and t_{Li} represent the tax rates levied on capital and labor, assumed to be uniform across sectors, in the i^{th} period, and Va_j represents the required inputs of value added, in real terms, to the

j^{th} sector. 1/ L_j, the cost-minimizing inputs of labor to sector j, is then derived as

$$(34)\quad L_j = \left(\frac{1+t_{Ki}}{1+t_{Li}}\right)\left(\frac{1-\alpha_j}{\alpha_j}\right)\left(\frac{p_{Ki}}{p_{Li}}\right) K_j$$

and the nominal value added, va_j, is given by

$$(35)\quad va_j(p) = p_{Ki}(1+t_{Ki})K_j + p_{Li}(1+t_{Li})L_j : \begin{array}{l} i = 1 \text{ if } j \leq N \\ i = 2 \text{ if } J > N \end{array}$$

Given this vector va(p) of nominal value added, intertemporal Leontief prices, $\bar{p}(p)$ may be calculated as

$$(36)\quad \bar{p}(p) = va(p)(I-A)^{-1}$$

where A is the Leontief matrix of production defined in equation (1). Thus, a set of 2N prices has been calculated that gives zero profit to each activity operating in each period, corresponding to the assumed prices for capital and labor. A complete set of intertemporal prices is now given for all intermediate and final goods, as well as capital, labor, and financial assets, so the consumer's maximization problem may be solved as in equations (29) through (32). In particular, total demand for the j^{th} intermediate and final good, xL_j may be derived as 2/

$$(38)\quad xL_j \equiv \sum_{k=1}^{K} x_j^k$$

where x_j^k is the k^{th} consumer's demand for intermediate or final good j, as in equation (29), and where the summation is taken over all K

1/ The interpretation of t_{Ki} is a profit tax levied upon capital, while t_{Li} may be thought of as an income tax that is collected at the source, that is, a withholding tax.

2/ Here xL_j is supposed to denote a Leontief good.

consumers. 1/ The vector of activity levels, z, of the 2N activities required to produce this level of demand may then be derived as

$$(39) \quad z \equiv (z_1, z_2) = (I-A)^{-1} xL$$

Let y_{Kj}, y_{Lj} be the requirements of column j for capital and labor, as derived in equations (33) and (34). The total requirements for capital and labor by private industry in periods 1 and 2 are then

$$(40) \quad y_{KP1} = \sum_{j=1}^{N} z_j y_{Kj}, \quad y_{LP1} = \sum_{j=1}^{N} z_j y_{Lj}$$

$$y_{KP2} = \sum_{j=N+1}^{2N} z_j y_{Kj}, \quad y_{LP2} = \sum_{j=N+1}^{2N} z_j y_{Lj}$$

The total taxes collected by the central government in each of the two periods may now be calculated. If T_i denotes the taxes collected in period i, then

$$(41) \quad T_1 = \sum_{j=1}^{N} t_j xL_j + t_{K1} y_{KP1} + t_{L1} y_{LP1}$$

$$T_2 = \sum_{j=N+1}^{2N} t_j xL_j + t_{K2} y_{KP1} + t_{L2} y_{LP1}$$

In addition, the government also uses capital and labor to produce public goods in each of the two periods. Suppose that the real quantity of these public goods is given by Q_1, Q_2. 2/ The government has a Cobb-Douglas production function as given in equation (6), and the cost-minimizing quantities of capital and labor, y_{KGi}, y_{LGi}, used by the government in producing Q_i, and the total cost to the government, G_i, of producing this quantity may be derived.

1/ Equation (29) contains no superscript k to denote the individual consumer in order to avoid confusing notation.

2/ It is supposed that the government sets expenditure targets for public goods in real terms, irrespective of the cost of inputs. A more realistic approach might include the level of public goods in the individual consumer's utility function. Such a system is, however, beyond the scope of this study.

$$(42) \quad y_{KGi} = \left\{ \frac{(1-\beta_i)}{\beta_i} \frac{p_{Ki}}{p_{Li}} \right\}^{(\beta_i - 1)} Q_i$$

$$y_{LGi} = \frac{(1-\beta_i)}{\beta_i} \frac{p_{Ki}}{p_{Li}} y_{KGi}$$

$$G_i \; = p_{Ki} y_{KGi} + p_{Li} y_{LGi}$$

The deficit of the central government in period 1, D_1, is then given by 1/

$$(43) \quad D_1 \equiv G_1 + p_{M1}\bar{B}_0 - T_1$$

so that if D_1 is negative, the government runs a surplus. In the case of a surplus, it is assumed that the surplus is paid out as transfer payments to consumers, 2/ but in the case of a deficit, financing operations must take place. In Feltenstein (1983) the assumption is made that the value of bond financing is a constant fraction of the value of the government deficit. Here, however, we must first make a connection between real interest rates and the level of the government's real expenditures. It is possible for a particular program of expenditures to be technologically feasible, in the sense that it does not require inputs of capital or labor beyond the capacities of the economy, yet, at the same time, to lead to a deficit representing a level of real debt greater than that which people will be willing to hold. In such a case, the result would be that the government would attempt to finance its deficit by issuing infinite amounts of money or bonds (or both), driving the corresponding prices to zero. In order to avoid this problem of unbounded money and bond supply functions, a functional relationship will be imposed between the real level of government expenditure and the instantaneous real interest rate and rate of inflation. Accordingly, define Q_i in the following way; let h_i be a continuous function and $\bar{Q}_i$ some fixed, target level of output of public goods.

1/ Taxes paid by the government to itself for the use of capital and labor in its own production processes have not been introduced. The distortion thus induced would tend to bias factor allocation toward the government. It would be possible to introduce taxes upon the government if it was desired to remove such distortions. There are, however, certain minor technical problems.

2/ These transfer payments are not identically equal to the sum of the transfer payments included in the consumer's budget constraints, although at equilibrium they will be.

$$(44) \quad Q_i = h_i(R_i, \pi_i) \quad : \quad R_i \equiv r_i - \pi_i \qquad i = 1, 2$$

$$Q_i = \bar{Q}_i \quad : \quad R_i < \bar{R}_i \ , \quad \pi_i < \bar{\pi}_i \ , \quad \bar{R}_i, \ \pi_i > 0$$

$$h_i(\bar{R}_i, \bar{\pi}_i) = \bar{Q}_i$$

$$\frac{\partial h_i}{\partial R_i} \ , \ \frac{\partial h_i}{\partial \pi_i} < 0 \quad : R_i > \bar{R}_i \ , \ \pi_i > \bar{\pi}_i$$

Thus, real output of public goods will be equal to the initial target $\bar{Q}_i$ if both the rate of inflation, π_i, defined in equation (14), as well as the real interest rate, R_i, the nominal rate, r_i, having been defined in equations (19, 20), are below corresponding target rates, as the real interest or inflation rates rise above the target rates, the level of real government output of public goods approaches 0. In Section III, a particular functional form will be specified for h_i, but it should be noted at this point that making Q_i inversely related to real interest rates and inflation will allow us to put an upper bound to the supply of both money and bonds in each of the two periods.

The resulting deficit, given by equation (43), is financed by a combination of money and bonds. The distribution of the financing is not important for our model, only that it be a continuous function of the deficit. Accordingly, let B_1 be a continuous function such that

$$(45a) \quad p_{B1} y_{B1} = B_1(G_1 - T_1) \quad : \quad 0 < B_1(G_1 - T_1) < D_1$$

$$B_1(0) = 0, \ B_1(G_1 - T_1) = 0, \ G_1 - T_1 < 0$$

Thus, the nominal value of bond financing is a continuous function of the nominal deficit, not including *debt repayment*, and *no sale of* bonds takes place if there is a surplus. The change in the supply of money, $\tilde{y}_{M1}$, is then given by

$$(45b) \quad p_{M1} \tilde{y}_{M1} = D_1 - B_1(G_1 - T_1)$$

so that debt repayment is made in money.

In period 2 the formation of the government deficit is somewhat different, since it must pay not only for its current consumption but also for its debt obligations incurred in period 1. Accordingly,

$$(46) \quad D_2 = G_2 + p_{M2}y_{BG1} - T_2$$

The issuance of money and bonds in period 2 may be defined as before, using a new function B_2 such that

$$(47a) \quad p_{B2}y_{BG2} = B_2(G_2-T_2) \quad : \quad 0 < B_2(G_2-T_2) < G_2 - T_2$$

$$B_2(0) = 0,\ B_2(G_2-T_2) = 0 : \quad G_2-T_2 < 0$$

$$(47b) \quad p_{M2}\tilde{y}_{M2} = G_2 - T_2 - B_2(G_2-T_2) + p_{M2}y_{BG1} = D_2 - B_2(G_2-T_2)$$

where B_2 is continuous. We would, of course, have $B_1 = B_2$ if the government chooses to maintain the same financing rule in period 2 as in period 1. 1/

As regards the private issuance of debt, it has been assumed that capital formation is carried out by the private sector and that it is fully financed by the sale of bonds, which are identical to the bonds sold by the government. Suppose, then, that the rate of return on capital in period i+1 is p_{Ki+1}. The total return on a quantity, H_i, of new capital that was built in period i and which comes on line in period i+1 is then $p_{Ki+1}H_i$. The present discounted value in period i, PDV_i, is

$$(48) \quad PDV_i \equiv \frac{p_{Ki+1}H_i}{1 + r_i} = \frac{p_{Ki+1}H_i}{1 + \frac{p_{Mi+1} - p_{Bi}}{p_{Bi}}} = \frac{p_{Bi}p_{Ki+1}H_i}{p_{Mi+1}}$$

1/ Thus, the interest obligations incurred by the government in period 1 are paid off in period 2 in units of money, rather than being rolled over in new bond sales. This form of payment is not essential to the model, but allows a simpler proof of the boundedness of the period 2 supply of bonds. A similar reasoning applies to equation (45b).

Accordingly, if C_{Hi} is the cost in period i of producing the quantity H_i of capital, assuming cost minimization, the investment activity must require as an intertemporal break-even condition:

$$(49)\quad C_{Hi} = PDV_i = \frac{p_{Bi}p_{Ki+1}H_i}{p_{Mi+1}}$$

In Section III it will be shown how the private sector's issuance of bonds is derived from this relationship. If the investment function H_i, defined in equation (3), exhibited constant returns to scale, it would not be possible to solve equation (49), as C_{Hi}/H_i would be constant.

The investment firm, having found a level of investment H_i such that H_i, C_{Hi} satisfy equation (49), then sets

$$(50)\quad p_{Bi}y_{Bi} = C_{Hi}$$

One final assumption will be made, namely, that the investment firm recognizes the finite supply of capital and labor, and bounds its level of investment accordingly. Thus, if $H_i = H_i(y_{KHi}, y_{LMi})$ then

$$(51)\quad y_{KH1} < \bar{K}_0 \quad , \quad y_{LH1} < \bar{L}_0$$

$$y_{KH2} < H_1(\bar{K}_0, \bar{L}_0) + (1-\delta)\bar{K}_0 \quad , \quad y_{LH2} < \bar{L}_0$$

This boundedness constraint will imply that it is possible for the owners of the investment firm to realize a positive profit on their investment if the rate of return in the future is sufficiently high relative to their debt obligations. In this case, the profits will accrue as income to the investors.

4. Excess demand functions and the conditions for intertemporal equilibrium

In order to demonstrate the existence of an intertemporal equilibrium, that is, a set of prices for all goods and financial assets at which markets clear in both periods, excess demand functions must first be derived. The approach used is based upon Feltenstein (1983), in which endogenous supplies of money and bonds, along with an endogenous

government deficit, are reflected by extra dimensions in the price simplex and, accordingly, additional elements in the excess demand function. Here, the fact that behavior in one period of the model is related to behavior in the other period, combined with the presence of the issuance of private debt, makes the construction of these excess demand functions somewhat more complicated.

The presence of an intertemporal input-output matrix allows the vector of excess demand functions to be confined to the space of prices corresponding to capital, labor, money, bonds, and transfer payments, indexed by their time period. Accordingly, given an arbitrary vector of prices p, the nominal value added per unit of output may be derived for each of the 2N sectors producing intermediate and final goods, as in equation (35). Equation (36) then gives Leontief prices for each of the two periods, and equations (38) and (39) give total demand for intermediate and final goods, along with the corresponding level of production required of each activity in the Leontief matrix. Equations (40) through (45) derive the total required inputs of capital and labor in each period by the private sector and the government.

The aggregate supplies of capital and labor in period i, $\tilde{y}_{Ki}$, $\tilde{y}_{Li}$, by the government and that part of the private sector producing intermediate and final goods may also be derived as

$$(52) \quad \tilde{y}_{K1} = y_{KP1} + y_{KG1}, \; \tilde{y}_{L1} = y_{LP1} + y_{LG1}$$

The total requirements of capital and labor in each period include also their usage in investment. Equations (3) and (43) determine, as shown in Section III, the inputs of capital and labor, y_{KHi}, y_{LHi}, required by the investment activity in period i, so the corresponding total requirements, $\hat{y}_{Ki}$, $\hat{y}_{Li}$, are given by

$$(53) \quad \hat{y}_{Ki} = y_{KHi} + \tilde{y}_{Ki}, \; \hat{y}_{Li} = y_{LHi} + \tilde{y}_{Li}$$

The total supplies of the capital and labor, y_{Ki}, Y_{Li}, are then

$$(54) \quad y_{K1} = -\hat{y}_{K1} + \bar{K}_0, \; y_{K2} = -\hat{y}_{K2} + (1-\delta)\bar{K}_0 + H_1$$

$$y_{L1} = -\hat{y}_{L1} + \bar{L}_0, \; y_{L2} = -\hat{y}_{L2} + \bar{L}_0$$

where $\bar{K}_0$, $\bar{L}_0$ are the aggregate initial stocks of capital and labor, summed over all consumers, and H_1 is the real level of investment in period 1.

The change in the money supply in period i, $\tilde{y}_{Mi}$, is given by equations (45b) and (47b), so that the total supply of money in each period, y_{Mi}, is

$$(55)\quad y_{M1} = \bar{M}_0 + \tilde{y}_{M1},\ Y_{M2} = Y_{M1} + \tilde{y}_{M2}$$

The supply of bonds in each period, y_{Bi}, is given by

$$(56)\quad y_{B1} = \bar{B}_0 + y_{BG1} + y_{Bp1}$$

$$y_{B2} = y_{B1} + y_{BG2} + y_{Bp2}$$

where y_{BG1}, the government's issuance of bonds in period 1, is given by equation (45a), y_{BG2} is given by equations (45a) and (47a), and y_{Bpi} is derived from equation (50).

An aggregate supply vector, y has now been derived, where

$$(57)\quad y \equiv (y_1,\ y_2) \equiv (y_{K1}, y_{L1}, y_{M1}, y_{B1}, y_{K2}, y_{L2}, y_{M2}, y_{B2})$$

As in Feltenstein (1983), this supply vector is augmented by two additional dimensions, corresponding to transfer payments in each of the two time periods. Accordingly, define y(p), the augmented supply vector, by

$$(58)\quad y(p) \equiv (y,\ \mu(D_1),\ \mu(D_2)):\quad \mu(D_i) = D_i\colon D_i < 0$$

$$\mu(D_i) = 0\colon D_i > 0$$

where D_i is the government deficit in period i.

The derivation of an augmented demand vector, $x(p)$, is now straightforward. Consumer demand for capital is zero, hence 1/

$$(59) \quad x_{Ki} \equiv 0 : \quad i = 1, 2$$

Equations (29) through (32) give individual demands for leisure, money, and bonds in each period, so adding across consumers gives the aggregate demands, x_{Li}, x_{Mi}, x_{Bi}. The aggregate demand vector, x, is then defined by

$$(60) \quad x \equiv (x_1, x_2) \equiv (0, x_{L1}, x_{M1}, x_{B1}, 0, x_{L2}, x_{M2}, x_{B2})$$

Finally, the augmented demand vector, $x(p)$, is defined by

$$(61) \quad x(p) \equiv (x, -TR_1, -TR_2)$$

where TR_i represents the proxy for government transfer payments that enters the consumer's maximization problem, as given in equation (26). The aggregate excess demand function, $u(p)$, is then defined as

$$(62) \quad u(p) \equiv x(p) - y(p)$$

so it must be shown that there exists some price p^*, where

$$(63) \quad p^* = (p^*_{K1}, p^*_{L1}, p^*_{M1}, p^*_{B1}, p^*_{K2}, p^*_{L2}, p^*_{M2}, p^*_{B2}, TR^*_1, TR^*_2)$$

such that $u(p^*) \leq 0$, that is, such that supply is equal to or greater than demand and that transfer payments received by consumers are equal to or greater than the amount actually paid out.

1/ This follows from the fact that consumers are assumed to satisfy their demand for savings entirely through the purchase of bonds.

III. The Existence of an Equilibrium

The proof of the existence of a competitive equilibrium in the model depends on demonstrating certain properties for the excess demand function defined in equation (62). It will first be shown that Walras' law holds at each period, and hence for the intertemporal excess demand. The value of supply in period 1, $S_1(p_1, p_2)$, is given by

$$(64) \quad S_1(p_1,p_2) = [\bar{p}(p_1)(I_1-A_1) - p_1 \cdot va(p_1)]z_1 - G_1 + p_{M1}\bar{M}_0 + p_{B1}\bar{B}_0 + p_{K1}\bar{K}_0 + p_{L1}\bar{L}_0 + p_{B1}y_{Bp1} - p_{K1}y_{KH1} - p_{L1}y_{LH1} + p_{M1}\tilde{y}_{M1} + p_{B1}y_{BG1}$$

where A_1 denotes the upper (or first period) quadrant of the intertemporal input-output matrix A. As,

$$(65) \quad p_{B1}y_{Bp1} = p_{K1}y_{KH1} + p_{L1}y_{LH1}$$

it follows that

$$(66) \quad S_1(p_1,p_2) = t_{K1}p_{K1}y_{Kp1} + t_{L1}p_{L1}y_{Lp1} - G_1 + p_{M1}\bar{M}_0 + p_{B1}\bar{B}_0 + p_{K1}\bar{K}_0 + p_{L1}\bar{L}_0 + p_{M1}\tilde{y}_{M1} + p_{B1}y_{BG1}$$

The value of demand in period 1, $E_1(p_1, p_2)$, is the value of the consumers' disposable income, minus that part of their income going to sales taxes, hence 1/

$$(67) \quad E_1(p_1,p_2) = p_{M1}\bar{M}_0 + p_{B1}\bar{B}_0 + p_{M1}\bar{B}_0 + p_{K1}\bar{K}_0 + p_{L1}\bar{L}_0 + TR_1 - \sum_{j=1}^{N} t_j xL_j$$

1/ Here, and in what follows, the K consumers have been aggregated in the model. The interested reader may easily carry out the aggregation to arrive at the equations presented.

Thus,

$$(68) \quad E_1 - S_1 = G_1 + p_{M1}\bar{B}_0 - (t_{K1}p_{K1}y_{KP1} + t_{L1}p_{L1}y_{LP1} + \sum_{j=1}^{N} t_j xL_j)$$

$$- p_{M1}\tilde{y}_{M1} - p_{B1}y_{BG1} + TR_1$$

$$= D_1 - p_{M1}\tilde{y}_{M1} - p_{B1}y_{BG1} + TR_1$$

If the first period components of x(p), y(p) are then denoted by $x_1(p)$, $y_1(p)$, as defined in equations (52) and (55), then

$$(69) \quad x_1(p) - y_1(p) = E_1 - S_1 - TR_1 - \mu(D_1)$$

$$= D_1 - p_{M1}\tilde{y}_{M1} - p_{B1}y_{BG1} - \mu(D_1)$$

$$= 0$$

as

$$D_1 > 0 \rightarrow D_1 = p_{M1}\tilde{y}_{M1} + p_{B1}y_{BG1}, \ \mu(D_1) = 0$$

$$D_1 < 0 \rightarrow \tilde{y}_{M1} = y_{BG1} = 0, \ \mu(D_1) = D_1$$

Thus, Walras' law holds in period 1.

In period 2 the value supply, $S_2(p_1, p_2)$, is given by

$$(70) \quad S_2(p_1,p_2) = [\bar{p}(p_2)(I_2-A_2) - p_2 \cdot va(p_2)]y_2 - G_2$$

$$+ p_{K2}(1-\delta)\bar{K}_0 + p_{L2}\bar{L}_0 + p_{M2}\mu(x_{B1}-y_{B1}) + p_{M2}x_{M1}$$

$$+ p_{B2}x_{B1} + p_{K2}H_1 + p_{B2}y_{Bp2} - p_{K2}y_{KH2} - p_{L2}y_{LH2}$$

$$+ p_{M2}\tilde{y}_{M2} + p_{B2}y_{BG2} + p_{B2}y_{Bp2} - p_{K2}y_{KH2} - p_{L2}y_{LH2}$$

The value of the second period initial stocks of money and bonds is taken to be equal to the values of the corresponding period 1 demands as an imposed equilibrium condition. Here, $\mu(x_{B1}-y_{B1}) \equiv x_{B1}-y_{B1}$: $x_{B1}-y_{B1} > 0$, $\mu(x_{B1}-y_{B1}) \equiv 0 : x_{B1}-y_{B1} < 0$. 1/ Thus,

$$(71)\quad S_2(p_1,p_2) = t_{K2}p_{K2}y_{Kp2} + t_{L2}p_{L2}y_{Lp2} - G_2 + p_{K2}(1-\delta)\bar{K}_0$$

$$+ p_{L2}\bar{L}_0 + p_{M2}\mu(x_{B1}-y_{B1}) + p_{M2}x_{M1} + p_{B2}x_{B1} + p_{K2}H_1$$

$$+ p_{M2}\tilde{y}_{M2} + p_{B2}y_{BG2} + p_{B2}y_{Bp2} - p_{K2}y_{KH2} - p_{L2}y_{LH2}$$

The value of demand in period 2 is, as before, given by the value of income minus tax payments.

$$(72)\quad E_2(p_1,p_2) = p_{K2}(1-\delta)\bar{K}_0 + p_{L2}\bar{L}_0 + p_{M2}\mu(y_{B1}-x_{B1})$$

$$+ p_{M2}x_{M1} + p_{M2}x_{B1} + p_{B2}x_{B1}$$

$$+ TR_2 - \sum_{j=N+1}^{2N} t_j xL_j$$

As in equation (71), $\mu(y_{B1}-x_{B1}) \equiv y_{B1}-x_{B1} : y_{B1}-x_{B1} > 0$, $\delta(y_{B1}-x_{B1}) \equiv 0 :$ $y_{B1}-x_{B1} < 0$. Hence,

$$(73)\quad E_2-S_2 = G_2-(t_{K2}p_{K2}y_{Kp2} + t_{L2}p_{L2}Y_{Lp2} + \sum_{j=N+1}^{2N} t_j xL_j)$$

$$+ p_{M2}\mu(y_{B1}-x_{B1})-p_{M2}\mu(x_{B1}-y_{B1})+p_{M2}x_{B1}-p_{K2}H_1-p_{M2}\tilde{y}_{M2}$$

$$- p_{B2}y_{BG2}+TR_2$$

1/ The period 2 initial stock of money will be augmented by $\mu(x_{B1}-y_{B1})$. Proof of the existence of equilibrium will ensure that $x_{B1} = y_{B1}$ at equilibrium, so that there will be no augmentation of period 2 supply. A similar augmentation of the value of demand is made in equation (66). Both of these augmentations are made in order to ensure that Walras' law holds at an arbitrary set of prices, and market clearing in period 1 will ensure that they will vanish in period 2.

$$= G_2-(t_{K2}p_{K2}y_{Kp2}+t_{L2}p_{L2}y_{Lp2}+\sum_{j=N+1}^{2N} t_j x_{Lj}) + p_{M2}y_{B1} - p_{K2}H_1$$

$$- p_{M2}\tilde{y}_{M2} - p_{B2}y_{BG2} + TR_2$$

By equation (49),

(74) $p_{M2}y_{B1} - p_{K2}H_1 = p_{M2}y_{B1} - p_{M2}y_{Bp1} = p_{M2}y_{BG1}$

Thus,

(75) $E_2 - S_2 = (G_2 + p_{M2}y_{BG1} - T_2) - p_{M2}\tilde{y}_{M2} - p_{B2}y_{BG2} + TR_2$

$$= D_2 - p_{M2}\tilde{y}_{M2} - p_{B2}y_{BG2} + TR_2$$

as $p_{M2}y_{BG1}$ represents the government's debt obligation in period 2.

Thus,

(76) $x_2(p) - y_2(p) = E_2 - S_2 - TR_2 - \mu(D_2) = 0$

as in equation (69). Thus, Walras' law also holds for the intertemporal excess demand function u(p) defined in equation (62).

It must now be shown that the intertemporal excess demand function u(p) defined in equation (63) is continuous and bounded in prices. To demonstrate this, it may be noted that:

Lemma 1: The value added to the j^{th} sector, $va_j(p)$ is a continuous function of p, the vector of intertemporal prices for capital, labor, and financial assets.

Thus,

Lemma 2: The intertemporal Leontief prices $\bar{p}(p)$ defined in equation (36) are continuous in p.

Accordingly, the levels of demand for the individual consumer, as defined in equations (31), are continuous in p. Summing over all consumers:

Lemma 3: The aggregate demand for the j^{th} intermediate or final good, xL_j, given by equation (31), is continuous in p, as are x_{Ki}, x_{Li}, x_{Mi}, x_{Bi}, the aggregate demands for capital, labor, money, and bonds in period i.

The activity levels, z, for the Leontief matrix A representing private production are thus also continuous in p by equation (39) so that:

Lemma 4: The inputs of capital and labor in each period to private production y_{Kpi}, y_{Lpi}, i = 1;2 are continuous in p.

The government's use of capital and labor in producing public goods must now be considered. The real interest rate, R_i, is given by

$$(77)\quad R_i \equiv \frac{p_{M(i+1)} - p_{Bi}}{p_{Bi}} - \frac{p_{Mi} - p_{M(i+1)}}{p_{M(i+1)}} = \frac{p_{M(i+1)}}{p_{Bi}} - \frac{p_{Mi}}{p_{M(i+1)}}$$

while the rate of inflation, π_i, is 1/

$$(78)\quad \pi_i \equiv \frac{p_{M(i-1)} - p_{Mi}}{p_{Mi}}$$

The rule for adjusting the real quantity of the output of public goods, Q_i, is then given by

$$(79)\quad Q_i = \bar{Q}_i \qquad \text{if } R_i < \bar{R}_i,\ \pi_i < \bar{\pi}_i$$

$$Q_i = \frac{\bar{Q}_i}{1 + R_i - \bar{R}_i} \qquad \text{if } R_i > \bar{R}_i,\ \pi_i < \bar{\pi}_i$$

1/ Thus, π, the rate of inflation in period 1, is dependent upon the price of money prior to period 1, $\bar{p}_{M0}$. This is exogenously given, and homogeneity in prices may be maintained if we extend the definition of intertemporal prices to p = ($\bar{p}_{M0}$, p_{K1}, p_{L1}, p_{M1}, p_{B1}, p_{K2}, p_{L2}, p_{M2}, p_{B2}). Here $\bar{p}_{M0}$ will not, however, enter as a variable price in the solution.

$$Q_i = \frac{\bar{Q}_i}{1 + \pi_i - \bar{\pi}_i} \qquad \text{if } R_i < \bar{R}_i,\ \pi_i > \bar{\pi}_i$$

$$Q_i = \min \left\{ \frac{\bar{Q}_i}{1 + R_i - \bar{R}_i},\ \frac{\bar{Q}_i}{1 + \pi_i - \bar{\pi}_i} \right\} \text{if } R_i > \bar{R}_i,\ \pi_i > \bar{\pi}_i$$

where $\bar{R}_i$ and $\bar{\pi}_i$ represent the real interest rate and rate of inflation, respectively, desired by the government in period 1. 1/ By equation (79) Q_i is easily seen to be continuous in p and hence, by equation (42) so are y_{KGi} and y_{LGi}, the government's inputs of capital and labor in period i. Thus:

Lemma 5: Government inputs of capital and labor in period i, y_{KGi}, y_{LGi}, respectively, as well as the cost of government production G_i, are continuous in p.

Lemmas (3-5) and equations (41) and (43-46) give:

Lemma 6: T_i, the tax revenues collected by the government in period i, and D_i, the government deficit in period i, are continuous in p.

By equations (44-47),

Lemma 7: y_{BGi}, $\tilde{y}_{Mi}$, the government's issuance of bonds and money in period i, are continuous in p.

1/ $\bar{R}_i$ might, for example, be taken to be equal to the long run real rate of interest, while $\bar{\pi}_i$ might be equal to a weighted average of past rates of inflation, as in an adaptive expectations framework. $\bar{R}_i$, $\bar{\pi}_i$ need only be finite constants for our results to hold.

Turning to private investment, from equations (3) and (49) it is seen that:

Lemma 8: H_i, the real quantity of capital produced in period i, is continuous in p.

Lemma (3), equations (49) and (70), and Lemma 7, give:

Lemma 9: y_{Bi}, the total supply of bonds in period i, is a continuous function of p.

Finally, by Lemmas (4-5), Lemmas (7-8) and equations (3) and (70):

Lemma 10: y_{Ki}, y_{Li}, y_{Mi}, the supply of capital, labor, and money, respectively, in period i, are continuous functions of p.

Equation (63) leads to:

Lemma 10a: u(p), the augmented excess demand function, is continuous in p.

The excess demand function u(p) must now be shown to be bounded. Because of the assumed bounds set on the individual consumer's demand for intermediate-final goods, as well as leisure, the aggregate demand for the j^{th} intermediate-final good, as in equation (38), is bounded also. Hence, by equations (39-40), is produced:

Lemma 11: y_{Kpi}, y_{Lpi}, the inputs of capital and labor to private industry in period i, are bounded.

By equation (41):

Lemma 12: T_i, the taxes collected by the government in period i, are bounded.

By equation (42) and the fact that $Q_i < \bar{Q}_i$:

Lemma 13: y_{KGi}, y_{LGi}, the inputs of capital and labor to government production in period i, are bounded, as is G_i, the government expenditure in period i, and D_i, the corresponding deficit.

Now $D_1 \equiv G_1 - T_1 < G_1$ and

$$(80)\quad G_1 = p_{K1}\left[\frac{(1 - B_1)}{B_1}\ \frac{p_{K1}}{p_{L1}}\right]^{(B_1 - 1)} \left[1 + \frac{(1 - B_1)}{B_1}\ \frac{p_{K1}}{p_{L1}}\right] Q_1$$

$$\equiv \lambda Q_1 :\ \lambda \equiv p_{K1}\left[\frac{(1 - B_1)}{B_1}\ \frac{p_{K1}}{p_{L1}}\right]^{(B_1 - 1)} \left[1 + \frac{(1 - B_1)}{B_1}\ \frac{p_{K1}}{p_{L1}}\right]$$

$$= \frac{\lambda p_{B1} p_{M2} \bar{Q}_1}{p_{B1}p_{M2} + p_{M2}^2 - p_{M1}p_{B1} - \bar{R}_1 p_{B1} p_{M2}} \quad \text{if } R_1 > \bar{R}_1$$

Thus, according to equation (45a)

$$y_{BG1} < \frac{\lambda\ p_{M2}\ \bar{Q}_1}{p_{B1}p_{M2} + p_{M2}^2 - p_{M1}p_{B1} - \bar{R}_1 p_{B1} p_{M2}}$$

where y_{BG1} is the government's issuance of bonds in period 1. Suppose that y_{BG1} is unbounded, so that $y_{BG1} \to \infty$. Then,

$$p_{B1}p_{M2} + p_{M2}^2 - p_{M1}p_{B1} - \bar{R}_1 p_{B1} p_{M2} \to 0$$

or

$$\frac{p_{M2}^2 - p_{M1}p_{B1}}{p_{B1}p_{M2}} \to \bar{R}_1 - 1$$

Thus,

$$R_1 \to \bar{R}_1 - 1 < \bar{R}_1$$

which contradicts the original assumption. We therefore must have $R_1 < \bar{R}_1$. But, in this case, $p_{B1} > 0$ by equation (77) and hence $y_{BG1} < \infty$.

Similarly, by equations (46-47), y_{BG2}, the issuance of government bonds in period 2, is also bounded, so that:

Lemma 14: The government's issuance of bonds in period i, y_{BGi}, is bounded.

A similar result needs to be demonstrated for the government's issuance of money. Now

$$
\begin{aligned}
p_{mi}\, \tilde{y}_{Mi} &= D_i - p_{Bi}\, y_{BGi} \\
&< D_i \\
&< \lambda\, Q_i \text{ where } \lambda \text{ is defined in equation (80).} \\
&= \lambda\, \bar{Q}_i \text{ if } p_{Mi}\, \frac{p_{M(i-1)}}{(1+\bar{\pi}_i)}
\end{aligned}
$$

Thus, in the first case y_{Mi} is clearly bounded, as $p_{Mi} > 0$, assuming $p_{M(i-1)} > 0$. In the second case

$$\tilde{y}_{Mi} = \frac{\lambda\, \bar{Q}_i}{p_{M(i-1)} - p_{Mi}\bar{\pi}_i}$$

which is also bounded, given the constraint on p_{Mi}. Thus:

Lemma 15: $\tilde{y}_{Mi}$, the government's issuance of money in period i, is bounded.

Let C_{Hi} denote the cost of private investment in period i. Then y_{Bpi}, the private issuance of bonds in period i, is given by:

$$y_{Bpi} = \frac{C_{Hi}}{p_{Bi}}$$

Thus, private debt obligations in period i+1 are given by

$$p_{Mi+1}\, y_{Bpi} = p_{Mi+1} \frac{C_{Hi}}{p_{Bi}}$$

so that as an equilibrium condition the result must be

$$p_{Mi+1} \frac{C_{Hi}}{p_{Bi}} = p_{Ki+1}\, H_i$$

Thus,

$$(81) \quad \frac{C_{Hi}}{H_i} = \frac{p_{Ki+1}\, p_{Bi}}{p_{Mi+1}}$$

Combining equation (81) with the cost minimization conditions for the investment function:

$$y_{KHi} = \left[\frac{p_{Ki+1}\, p_{Bi}}{p_{Mi+1}} \; \frac{p_{Ki}^{\beta_i - 1} \; p_{Li}^{-\beta_i}}{(1 + \beta_i/\alpha_i)} \; \frac{\beta_i}{\alpha_i}^{\beta_i} \right]^{\frac{1}{1 - \alpha_i - \beta_i}}$$

$$y_{LHi} = \frac{\beta_i}{\alpha_i} = \frac{p_{Ki}}{p_{Li}} \quad y_{KHi}$$

Now by assumption, $y_{KH1} < \bar{K}_o$, $y_{LH1} < \bar{L}_o$. Let $\bar{H}_1 = \max \left\{ y_{KH1}^{\alpha_1}\, y_{LH1}^{B_1} : y_{KH1} < \bar{K}_o,\ y_{LH1} < \bar{L}_o \right\}$. Then $y_{KH2} \leq (1-\delta)\bar{K}_o + \bar{H}_1$, $y_{LH2} < \bar{L}_o$ so that $C_{Hi} = p_{Ki}\, y_{KHi} + p_{Li}\, y_{LHi} < \bar{C}_i$ for some $\bar{C}_i < \infty$. In order to show that y_{Bpi}, the private sector's issuance of bonds in period i, is bounded, it needs only to be shown that

$$\lim_{p_{Bi} \to 0} \frac{C_{Hi}}{p_{Bi}}$$

is bounded. But

$$\frac{C_{Hi}}{p_{Bi}} = \left(1 + \frac{\beta_i\ p_{Ki}}{\alpha_i\ p_{Li}}\right)\left[\frac{p_{Ki+1}}{p_{Mi+1}} \quad \frac{p_{Ki}^{B_i - 1}\ p_{Li}^{-B_i}}{(1 + B_i\alpha_i)} \quad \frac{\beta_i}{\alpha_i}^{B_i}\right]^{\frac{1}{1 - \alpha_i - \beta_i}} p_{Bi}^{\frac{\alpha_i + \beta_i}{1 - \alpha_i - \beta_i}}$$

so that

$$\lim_{p_{Bi} \to 0} \frac{C_{Hi}}{p_{Bi}} = 0$$

Thus:

<u>Lemma 16</u>: y_{Bpi}, the private sector's issuance of bonds in period i, is bounded.

Consider now x^j_{Bi}, the j^{th} consumer's demand for bonds in period i. If $\bar{y}_{BGi}$, $\bar{y}_{Bpi}$ are the upper bounds on the government's and the private sector's issuance of bonds in period i, a bound already shown to exist, then, by assumption

$$x^j_{B1} < \bar{y}_{B1} \equiv \bar{y}_{BG1} + \bar{y}_{Bp1}$$

so that: $$x_{B2} = \sum_{j=1}^{J} x^j_{B1} < J\ \bar{y}_{B1}$$

where $J > 0$ is the number of consumers in the economy.

Hence, the aggregate first period demand for bonds is bounded.

Now, in period 2, supply of bonds, y_{B2}, is given by:

$$y_{B2} = x_{B1} + y_{BG2} + y_{Bp2}$$

$$< J\ \bar{y}_{B1} + \bar{y}_{BG2} + \bar{y}_{Bp2} \equiv \bar{y}_{B2}$$

Thus, y_{B2} is bounded. It has also been assumed:

$$x^j_{B2} < \bar{y}_{B2}$$

so that: $x_{B2} = \sum_{j=1}^{J} x^j_{B2} < J\,\bar{y}_{B2}$

Thus:

Lemma 17: x_{Bi}, y_{Bi}, the aggregate demand for and supply of bonds, respectively, in period i, are bounded.

Lemma 15 and the assumption that $x^j_{M1} < \bar{M}_o + \bar{y}_{M1}$ gives:

$$x_{M1} = \sum_{j=1}^{J} x^j_{M1} < J(\bar{M}_o + \bar{y}_{M1})$$

and: $y_{M2} = x_{M1} + \tilde{y}_{M2} < J(\bar{M}_o + \bar{y}_{M1}) + \bar{y}_{M2}$

where $\bar{y}_{Mi}$ are the upper bounds on issuance of money in period i. Hence, by assumption,

$$x_{M2} \equiv \sum_{j=1}^{J} x^j_{M2} < J\left[J(\bar{M}_o + \bar{y}_{M1}) + \bar{y}_{M2}\right]$$

and thus:

Lemma 18: x_{Mi}, aggregate demand for money in period i, is bounded.

Finally, by Lemmas (12)-(14) and equations (43) and (46):

Lemma 19: D_i, the deficit (if positive) or surplus (if negative) of the government in period i is bounded.

Recalling the augmented excess demand function u(p) defined in equation (62) leads to the conclusion:

Lemma 20: u is a nonempty, bounded, continuous function of p.

A standard line of reasoning, depending upon Brouwer's fixed point theorem, may now be applied to demonstrate the required result, namely:

Theorem: 1/ There exists a price vector p^*, as in equation (63), such that $u(p^*) \leq 0$.

IV. Conclusion

A computational general equilibrium model has been constructed that is designed to analyze the crowding out of the private sector by the public sector. The model is fully dynamic, having two periods, plus a past (before the first period) and a future (after the second period). Both consumers and firms have perfect foresight for the two periods in question, so that prices, tax liabilities, and transfers received from the government in period 2 are correctly forecast in period 1. Private enterprises are constrained to cover current expenditures from current revenues, while investment is financed by the sale of bonds. The government, on the other hand, is not required to cover the cost of its production of public goods, whose quantity it sets in real terms. When it incurs a deficit, the government issues a combination of money and bonds to cover its loss, but a surplus, if it occurs, is distributed to consumers. Perfect foresight precludes the possibility of risk, so that private and public bonds are perfect substitutes, from the point of view of the consumer. The equilibrium conditions on public and private debt are quite different, however, since capital produced by private investment in period 1, and, hence, coming on line in period 2, must yield a return in period 2 equal to the debt obligations on the bonds that financed it. The government, on the other hand, must add the debt obligations incurred in period 1 and coming due in period 2 to its current expenditures in that period.

There are a number of directions for future research. First and most obvious would be the empirical implementation of the model. Fullerton and others (1981) have constructed the real side of a computational general equilibrium model, applied to the United States, while Jorgenson (1983) has estimated value-added functions of the form that

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1/ See, for example, Shoven (1974) for a proof of this result. Brouwer's fixed point theorem may be used, rather than the usually invoked Kakutani's theorem since a single valued function, u, has been defined.

the model presented in this paper requires on an industry-by-industry basis. A number of studies have estimated investment functions for the United States, 1/ so as a first step these results could be used until new estimations are made. The main task, then, would be to estimate the financial side of the model, assuming it is desired to apply it to the United States.

This estimation is not so overwhelming as it seems, since if a nominal-demand-for-money function is estimated and combined with the estimates of Fullerton and others (1981) of consumer demand for goods, the demand for bonds, equivalent to savings in this model, can be derived as a residual. The fully estimated model might then be simulated to analyze, for example, the extent to which the crowding out of the private sector is affected by different rates of consumer time preference, a parameter that we cannot directly estimate. The policymaker interested in financing deficits by means of increased public indebtedness may, as a secondary concern, wish to examine the impact that his choice of policies has upon interest rates and private output, variables that will be generated by our model. If it is felt that the degree of crowding out that the model predicts is excessive, the possibility of cutting the deficit by raising individual tax rates could also be studied, and, simultaneously, the impact that the new tax regime would have on the welfare of different consumer groups could be calculated. There are, of course, many other policy simulations that could be carried out with an empirical version of the model.

On the theoretical side, the model could be extended to include a foreign trade sector with an endogenous balance of payments, allowing for foreign borrowing to finance private investment and government deficits, but if it seems desirable to incorporate a floating exchange rate, it would be necessary to develop a theory of exchange rate determination within our framework. Such a theory would require a notion of risk and portfolio decision in exchange markets and would represent a considerable advance over our current system. 2/ On the domestic side, an important innovation would be to introduce an overlapping generations structure to the model. Because the old and the new generations would be discounting the future stream of returns on capital over different time horizons, such a structure would permit the existence of a secondary market for capital. Finally, for certain technical reasons, capital gains taxation has not been allowed. A significant improvement in the model would be achieved by introducing such taxes, since the consumer's attitude toward bond purchases would then be directly affected by the tax regime.

1/ See, for example, Jorgenson and Stephenson (1969), Christensen and Jorgenson (1970), and Jorgenson (1971) and (1974).

2/ Feltenstein (1984) considers a system of managed floating but avoids the issue of risk.

References

Aghevli, B.B., and M.S. Khan, "Government Deficits and the Inflatonary Process in Developing Countries," Staff Papers, International Monetary Fund (Washington), Vol. 25 (September 1978), pp. 383-416.

Blinder, Alan S., and Robert M. Solow, "Does Fiscal Policy Matter?," Journal of Public Economics, Vol. 2 (November 1973), pp. 319-37.

_____, "Analytical Foundations of Fiscal Policy," in The Economics of Public Finance (The Brookings Institution, Washington, 1974), pp. 3-115.

Brunner, Karl, and Allan H. Meltzer, "Money, Debt, and Economic Activity," Journal of Political Economy, Vol. 80 (September-October 1972), pp. 837-51.

Buiter, Willem H., "'Crowding Out' and the Effectiveness of Fiscal Policy," Journal of Public Economics, Vol. 7 (June 1977), pp. 309-28.

Christ, Carl F., "A Simple Macroeconomic Model with a Government Budget Restraint," Journal of Political Economy, Vol. 76 (January/February 1968), pp. 53-67.

Christensen, Laurits R., and Dale W. Jorgenson, "U.S. Real Product and Real Factor Input, 1929-1967," Review of Income and Wealth, Vol. 16 (March 1970), pp. 19-50.

Clements, Kenneth W., "A General Equilibrium Econometric Model of the Open Economy," International Economic Review, Vol. 21 (June 1980), pp. 469-88.

Cohen, Darrel, and J. Stuart McMenamin, "The Role of Fiscal Policy in a Financially Disaggregated Macroeconomic Model," Journal of Money, Credit and Banking, Vol. 10 (August 1978), pp. 322-36.

Debreu, Gerard, Theory of Value: An Axiomatic Analysis of Economic Equilibrium (Wiley, New York, 1959).

Dervis, Kemal, Jaime DeMelo, and Sherman Robinson, General Equilibrium Models for Development Policy (Cambridge University Press, London, 1982).

Feltenstein, Andrew, "A General Equilibrium Approach to the Analysis of Trade Restrictions, with an Application to Argentina," Staff Papers, International Monetary Fund (Washington), Vol. 27 (December 1980), pp. 749-840.

Feltenstein, Andrew, "A Computational General Equilibrium Approach to the Shadow-Pricing of Trade Restrictions and the Adjustment of the Exchange Rate," Journal of Policy Modeling, Vol. 5 (No. 3, November 1983), pp. 333-61.

_____, "Private Investment, Public Deficits, and Crowding Out Within an Intertemporal General Equilibrium Model" (DM/83/1).

_____, "Money and Bonds in a Disaggregated Open Economy," in Applied General Equilibrium Analysis, ed. by H.E. Scarf and J.B. Shoven (Cambridge University Press, 1984).

Friedman, Benjamin M., "Crowding Out or Crowding In? Economic Consequences of Financing Government Deficits," Brookings Papers on Economic Activity (No. 3, November 1978), pp. 593-641.

Fullerton, Don, "On the Possibility of an Inverse Relationship Between Tax Rates and Government Revenues," Journal of Public Economics, Vol. 19 (October 1982 a), pp. 3-22.

_____, "Transition Losses of Partially Mobile Industry-Specific Capital," Quarterly Journal of Economics (1982 b).

_____, and others, "Corporate Tax Integration in the United States: A General Equilibrium Approach," American Economic Review, Vol. 71 (September 1981), pp. 677-91.

Gramlich, Edward M., "The Usefulness of Monetary and Fiscal Policy as Discretionary Stabilization Tools," Journal of Money, Credit and Banking, Vol. 3 (May 1971), pp. 506-32.

Grandmont, Jean-Michel, "Temporary General Equilibrium Theory," Econometrica, Vol. 45 (April 1977), pp. 535-72.

_____, and Guy Laroque, "On Money and Banking," Review of Economic Studies, Vol. 42 (April 1975), pp. 207-36.

Harberger, Arnold C., "The Incidence of the Corporation Income Tax," Journal of Political Economy, Vol. 70 (June 1962), pp. 215-40.

_____, "Efficiency Effects of Taxes on Income from Capital," in Effects of Corporation Income Tax, ed. by Marian Kreyzaniak (Wayne State University Press, Detroit, 1966).

Infante, Ettore F., and Jerome L. Stein, "Does Fiscal Policy Matter?," Journal of Monetary Economics, Vol. 2 (April 1976), pp. 473-500.

Jorgenson, Dale W., "Econometric Models for General Equilibrium Analysis," in Applied General Equilibrium Analysis, ed. by H.E. Scarf and J.B. Shoven (Cambridge University Press, 1983).

_____, "Econometric Studies of Investment Behavior: A Survey," Journal of Economic Literature, Vol. 9 (December 1971), pp. 111-47.

_____, "Investment and Production: A Review," in Frontiers of Quantitative Economics, ed. by Michael D. Intriligator and D. Kendrick, Vol. 2 (North-Holland Publishing Company, Amsterdam, 1974), pp. 341-75.

_____, and James A. Stephenson, "Issues in the Development of the Neoclassical Theory of Investment Behavior," Review of Economics and Statistics, Vol. 51 (August 1969), pp. 346-53.

Meyer, Lawrence H., "The Balance Sheet Identity, the Government Financing Constraint, and the Crowding-Out Effect," Journal of Monetary Economics, Vol. 1 (January 1975), pp. 65-78.

Miller, Marcus H., and John E. Spencer, "The Static Economic Effects of the U.K. Joining the EEC: A General Equilibrium Approach," Review of Economic Studies, Vol. 44 (February 1977), pp. 71-93.

Modigliani, Franco, and Albert Ando, "Impacts of Fiscal Actions on Aggregate Income and the Monetarist Controversy: Theory and Evidence," in Monetarism, ed. by Jerome L. Stein (North-Holland Publishing Company, Amsterdam, 1976), pp. 17-42.

Piggott, J., and J. Whalley, Economic Effects of U.K. Tax-Subsidy Policies: A General Equilibrium Appraisal (MacMillan, 1983).

Shoven, John B., "The Incidence and Efficiency Effects of Taxes on Income from Capital," Journal of Political Economy, Vol. 84 (December 1976), pp. 1261-83.

_____, "A Proof of the Existence of a General Equilibrium with Ad Valorem Commodity Taxes," Journal of Economic Theory, Vol. 8 (1974), pp. 1-25.

_____, "Applied General Equilibrium Tax Modeling" (unpublished manuscript, 1982).

_____, and John Whalley, "A General Equilibrium Calculation of the Effects of Differential Taxation of Income from Capital in the U.S.," Journal of Public Economics, Vol. 1 (November 1972), pp. 281-321.

_____, "General Equilibrium with Taxes: A Computational Procedure and an Existence Proof," Review of Economic Studies, Vol. 40 (October 1973), pp. 475-89.

Slemrod, J., "A General Equilibrium Model of Taxation with Endogenous Financial Behavior," NBER Working Paper No. 799 (November 1981).

Spencer, Roger W., and William P. Yohe, "The 'Crowding Out' of Private Expenditures by Fiscal Policy Actions," Federal Reserve Bank of St. Louis Review, Vol. 52 (October 1970), pp. 12-24.

Tanzi, Vito, "Inflation, Real Tax Revenue, and the Case for Inflationary Finance: Theory with an Application to Argentina," Staff Papers, International Monetary Fund (Washington), Vol. 24 (September 1978), pp. 417-51.

Tobin, James, and Willem H. Buiter, "Long-Run Effects of Fiscal and Monetary Policy on Aggregate Demand," in Monetarism, ed. by Jerome L. Stein (North-Holland Publishing Company, Amsterdam, 1976), pp. 273-309.

Whalley, John, "A General Equilibrium Assessment of the 1973 United Kingdom Tax Reform," Economica, Vol. 42 (May 1975), pp. 139-61.

_____, "The United Kingdom Tax System 1968-1970: Some Fixed Point Indications of its Economic Impact," Econometrica, Vol. 45 (November 1977), pp. 1837-58.

_____, "An Evaluation of the Recent Tokyo Round Trade Agreement through a General Equilibrium Model of World Trade Involving Major Trading Areas," Journal of Policy Modeling (1982).