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Private Investment, Public Deficits, and Crowding Out Within an Intertemporal General Equilibrium Model

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I. Introduction

The aim of this paper will be to construct an economic model that is designed to estimate the extent to which public spending crowds out private production and capital formation. Although the analysis presented here is purely theoretical, it should be seen as a first step in the building of an empirical model to be applied, specifically, to the United States. 2/ The approach taken here is especially useful for policy analysis, since it simultaneously allows the consideration of disaggregated fiscal measures, such as changes in individual tax rates or transfer payments; yet at the same time it incorporates the macroeconomic aspects of fiscal policy, such as the rules for deficit financing, and the interaction between government deficits, interest rates, and inflation. In addition, by dissagregating the private sector, a comparison can be made of the relative extent to which individual industries suffer (or benefit) from public sector spending policies. Although, as with all economic models applied to real situations, a certain degree of skepticism is required in accepting the results derived from simulations of this model, the estimation requirements are not significantly greater than those in a number of currently existing and accepted models. 3/

1/ I would like to thank Mario Blejer, Willem Buiter, Mohsin Khan, Alessandro Penati, Kenneth Rogoff, John Shoven, and Vito Tanzi for many helpful comments and criticisms. The errors remain, as always, my own.

2/ This application is motivated by the fact that, as shall be discussed shortly, a large amount of the required data work for the United States has already been carried out. It would be quite possible, however, to apply this model to a number of other industrial and less developed countries.

3/ See Shoven (1982) for a survey of some of these models.

Thus the policy conclusions resulting from the model should, at the very least, offer useful guidance to policymakers.

Before turning to a description of the model, it may be useful to briefly review certain aspects of the current literature on crowding out, so as to point out the differences of the model. The issue of crowding out has usually been examined in two different but related contexts. In the first of these, the public sector purchases large quantities of goods and finances these purchases either by taxes or by borrowing. Insofar as these purchases are used for the production of public goods, they will no longer be available as inputs to the production of the private sector, the output of which will therefore be forced to decline. The second such context, usually referred to as financial crowding out, concerns the government's increasing its borrowing requirements, thereby driving up the interest rate. Access to credit markets is thus made more expensive to the private sector, so that it is forced to curtail that part of its capital formation that is not self-financed. Indirect crowding out may also occur as the rising interest rate may cause current consumption, and hence demand for the output of the private sector, to fall.

Financial crowding out has traditionally been analyzed within the context of macroeconomic models in which the private sector is aggregated into a single unit, private capital formation is dependent upon the interest rate, and the interest rate is, in turn, dependent upon the government's borrowing requirements and, hence, its deficit. 1/ But there are severe limitations to this aggregative approach. Borrowing requirements are different across industry, so one would expect the government's borrowing to have a differential impact on the private sector. The aggregation of demand also precludes any analysis of the relative impact of government fiscal policies on the welfare of different consumer groups. In addition, the models are usually only valid for small changes, so it is difficult, if not impossible, to estimate the impact of sudden, rapid increases in government borrowing. Often governments attempt to simultaneously increase tax revenue and borrowing. Because the macroeconomic models in question do not normally separate tax revenues and government expenditure, such policies cannot be properly dealt with. 2/ Despite such limitations, macroeconomic models of this type have been used widely to give policy advice, often in circumstances where their underlying assumptions cause them to not be strictly valid.

1/ Among such models are those of Blinder and Solow (1973, 1974), Brunner and Meltzer (1972), Buiter (1977), Christ (1968), Cohen and McMenamin (1978), Friedman (1978), Gramlich (1971), Infante and Stein (1976), Meyer (1975), Modigliani and Ando (1976), Spencer and Yohe (1970), and Tobin and Buiter (1976).

2/ See Tanzi (1978) and Aghevli and Khan (1978) for models which do distinguish between taxes and revenues.

The question of resource crowding out is increasingly being examined within the framework of computational general equilibrium (CGE) models of taxation. Such models, originally inspired by the work of Harberger (1962, 1966) on tax incidence, have been developed in Shoven and Whalley (1972, 1973), Shoven (1976), Fullerton (1982 a, 1982 b), Fullerton and others (1981), Miller and Spencer (1977), Piggott and Whalley (1983), and Whalley (1975, 1977, 1982), among others, to examine incidence and welfare implications of changes in tax regimes. The advantages of these models, as compared with the macroeconomic ones, have been discussed at length in Shoven (1982); among them are the ability to deal with large changes in government policies, with disaggregated taxes, and with the analysis of the welfare implications of taxation via the examination of individual consumer categories. There are, however, a number of disadvantages. These models have been almost exclusively "real," so that, in particular, the public sector is constrained to have a balanced budget, owing to the absence of financial assets that could be used to finance a deficit. Because there is no money, and hence no price level or interest rate, it is impossible to analyze financial crowding out. From the point of view of the government policymakers, the advice given by such models must be quite suspect: if their results are to be believed, it is then necessary to believe that government deficits have no real impact; if they did then the balanced-budget results produced by the models would be meaningless.

Finally, neither type of model can adequately cope with the crowding out that is caused by government transfers to loss-making enterprises. The macroeconomic models are not able to cope because they fail to disaggregate the private sector, and the CGE models cannot cope because they require that such transfers be covered by increased taxes, rather than increased deficits, as is normally the case.

Research in which certain types of CGE models are expanded to include financial assets has recently been carried out by several authors. Clements (1980) allows for domestic credit expansion in a model of the United States, although it is exogenous with respect to public sector expenditure and revenues. Feltenstein (1980), in a model of Argentina, permits the existence of domestic and foreign financial assets, the endogeneity of the supply of which is dependent upon the balance of payments, while Feltenstein (forthcoming 1983) has an endogenous government deficit and corresponding financing via the issuance of money and bonds. Another approach has been that of Slemrod (1981), who constructs a CGE model that incorporates portfolio choice by consumers. For the policymaker the major flaw in these models is that they do not permit both endogenous public deficits and private investment, and therefore cannot adequately cope with the issue of crowding out. We will present here a model, intended to address this flaw, which has a computational general equilibrium structure, but which also has considerable macroeconomic content. The model is dynamic; it has two periods and the notion of a past (before period 1) and a future (after period 2). Both consumers and firms have perfect foresight for the two periods, so that the prices, tax liabilities, and transfers received from the government in period 2 are correctly anticipated in period 1. 1/ In the future (after period 2) consumers become perfectly myopic, expecting the same structure of prices, taxes, and government transfers to prevail then as in period 2. 2/

Firms in the private sector are constrained to cover current expenditures by current revenue, while capital formation is financed by the sale of bonds. The government, on the other hand, sets its program of expenditure in real terms and is not required to cover costs from tax revenues, and when it incurs a deficit, the government issues a combination of money and bonds to cover its loss. Consumers are required to hold money to cover transaction costs, and they purchase bonds in order to save for the future. Perfect foresight precludes the possibility of risk, so that private and government bonds are viewed as being identical by the consumer. The equilibrium condition on privately issued debt is that new capital produced in period 1, which comes on line in period 2, must yield a return in period 2 equal to the obligations on the bonds that financed it. The government, on the other hand, must add the debt obligations incurred in period 1 and coming due in period 2 to its current expenditures in that period.

Intuition tells us that consumers, who are disaggregated, maximize intertemporal utility functions and derive a demand for bonds as a method of savings. Simultaneously, the government, when it runs a deficit, sells bonds at a discount, which becomes greater the larger the deficit is. As the corresponding interest rate becomes higher, consumers satisfy the Fisherian relation and shift their consumption to the future, releasing resources to the government. These resources, in particular savings, are increasingly unavailable to the private sector, which is also constrained by the fact that the debt obligations it is incurring for period 2 are rising relative to the anticipated rate of return on its investment in period 1. The private sector thus suffers from both resource and financial crowding out. 3/

1/ The model may thus be interpreted as generating a rational expectations equilibrium, in which consumers have no incentive to revise their expectations of the future, having correctly anticipated period 2. The minimum length of time needed to introduce a dynamic framework is two periods, but there would be no difficulty in extending the model to several periods.

2/ This "closure" rule is made for purely technical reasons. We must allow for some future after the final period in order to avoid the requirement that in that final period there be a balanced government budget and no private investment, as consumers would, in the absence of a future, refuse to hold debt.

3/ We should emphasize, however, that our model does not yield a mechanical one-to-one correspondence between public deficits and crowding out, because the rising interest rate will not only have the abovementioned effects but also will increase the overall level of savings.

Our model includes profit, income, and sales taxes and allows for direct transfer payments by the government to consumers. Although we have not done so at this stage, the model could be expanded to include transfer payments to enterprises so that the third type of crowding out, caused by incorrect allocation of funds, could also be analyzed. The price level is endogenous, so that the inflationary impact of various government policies may be analyzed. There are also savings and investment functions, with the level of investment being driven by the demand for savings. The model would therefore lend itself to empirical implementations, as such functions, along with that representing the production technology, are commonly estimated.

Section II will present a formal description of our model, and Section III will be a conclusion, indicating certain directions for empirical implementation of the model and for future research.

II. The Model

1. Production

The structure of production is Leontief in intermediate and final production, while value added is produced by smooth production functions. 1/ Because the model incorporates perfect foresight in both production and consumption, production may be represented by a block-diagonal matrix, whose components refer to goods that are different in their dating. 2/ If goods $i = 1, \ldots, N$ refer to goods produced in period 1 and goods $N + 1, \ldots, 2N$ refer to goods produced in period 2, then the structure of the production matrix for intermediate and final goods is

$$(1) \quad \begin{matrix} a_{11}, & \ldots, & a_{1N}, & 0, & \ldots, & \ldots, & 0 \\ \cdot & & & & & & \cdot \\ \cdot & & & & & & \cdot \\ a_{N1}, & \ldots, & a_{NN}, & 0, & \ldots, & \ldots, & 0 \\ 0, & \ldots, & 0, & a_{N+1,N+1}, & \ldots, & & a_{N+1,2N} \\ \cdot & & & & & & \cdot \\ \cdot & & & & & & \cdot \\ \cdot & & & & & & \cdot \\ 0, & \ldots, & 0, & a_{2N,N+1}, & \ldots, & & a_{2N,2N} \end{matrix}$$

The upper block of the matrix refers to first period production, and the lower block refers to second period production. If there is no technological change between the two periods then the coefficients in the two blocks would be identical. Corresponding to each activity there

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1/ This formulation is used because of our eventual goal of an empirical application and has been described in greater detail in, for example, Fullerton and others (1981) and Feltenstein (1980).

2/ See Debreu (1959) for a discussion of the use of dated commodities.

is a continuous function $f_j(K_i, L_i)$, which produces value added for the j^{th} activity using capital and labor from the corresponding stocks that exist in period i. In order to be specific, assume that the value-added functions are Cobb-Douglas, hence of the form

$$(2) \quad f_j(K_i, L_i) = K_i^{\alpha_j} L_i^{(1-\alpha_j)}$$

In addition, there are investment activities, $F_i(K_i, L_i)$, which operate in period i, using inputs of capital and labor existing in that period, and which produce capital goods for period i + 1. 1/ The investment is considered to be part of the private sector, and since the capital that is produced only becomes available in the next period, the investment firm must pay for the input costs of its production in the current period but will receive the revenue from that capital in the next period only. 2/ Assume that the investment functions have constant elasticity of substitution form, hence

$$(3) \quad F_i(K_i, L_i) \equiv \left(b_{1i}^{1/s_i} K_i^{(s_i-1)/s_i} + b_{2i}^{1/s_i} L_i^{(s_i-1)/s_i}\right)^{s_i/(s_i-1)}$$

Capital in period 2 is then given by the initial capital stock plus whatever new capital has been produced in period 1. 3/ If $\overline{K}_0$ is the initial stock of capital at the beginning of period 1, and $\overline{L}_0$ is the initial stock of labor, then

$$(4) \quad K_2 = \overline{K}_0 + F_1(\overline{K}_0, \overline{L}_0)$$

$$K_f = K_2 + F_2(K_2, \overline{L}_0)$$

where K_2 is the stock of capital at the beginning of period 2, and K_f is the capital stock existing in the future (after period 2). 4/

1/ The investment function could also require intermediate and final goods as inputs, but for simplicity of exposition it will require only capital and labor as inputs.

2/ It would be possible to have investment activities distinguished by firms if we also had firm-specific capital, as in Fullerton (1982), and Dervis, DeMelo, and Robinson (1982), but to do so would not qualitatively change the nature of the model.

3/ Ignore depreciation, as the model is intended to reflect only a relatively short period of time. Also, for simplicity of exposition, we would prefer not to have to distinguish between gross and net investment.

4/ To avoid introducing a differential age structure into the model, assume zero growth in the population.

The government also produces public goods, for which it receives no revenue, via a smooth production function that uses capital and labor of the current period as inputs. 1/ Let G_i (K_i, L_i) denote this function in period i and for simplicity also assume the function to be Cobb-Douglas, hence of the form

$$(5) \quad G_i (K_i, L_i) = K_i^{\beta_i} L_i^{(1-\beta_i)}$$

The government is assumed to decide, at the beginning of period i, on the level of output of public goods, in real terms, to be produced during the period. The government then solves the equation

$$(6) \quad \overline{G}_i = K_i^{\beta_i} L_i^{(1-\beta_i)}$$

where $\overline{G}_i$ is the real quantity of public goods to be produced in period i, in such a way as to minimize the cost of production. The financing of the cost of this production will be discussed in Section II.3, but note here that the government issues money and bonds, which are also sold by private investment activity.

2. Consumption

The consumers in the model are viewed as living for the entire period of the model, namely, for the two periods being solved and also for the third, or future period. Since they may have initial endowments of goods other than labor, it is implicitly supposed that they were alive before period 1, so that their holdings of capital and financial assets may be carried over into period 1. In periods 1 and 2 the consumers have perfect foresight, that is, they perfectly anticipate all prices of period 2 while they are still in period 1. 2/ They also correctly anticipate their tax obligation (or transfer payments received) in period 2. In the future (after period 2) the consumers become perfectly myopic, by which we mean that they anticipate the same relative prices and tax

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1/ Rather than having the government operate its own production function, it would also be possible to have the government buy directly from the private sector. Introducing a government production function allows us, however, to represent directly changing public policy toward the relative importance of hiring capital or labor. If, for example, the government wished to increase employment, it could, in the model, change the weights given to capital and labor in its production function.

2/ A rational expectations equilibrium is being defined in which consumers' expectations of period 2 are perfectly fulfilled, so that they have no incentive to revise these expectations in the future. If the model contained more than two periods, it would be quite possible that information available for the time period after period 2 might be used to determine the consumers' choices in periods 1 and 2.

obligations (or transfer payments) will hold in the future as held in period 2. These obligations and prices will simply be scaled up by whatever the anticipated rate of inflation is. An interpretation of this type of expectation is that after having been proven to be correct in period 2, the consumers believe that the economy is on a steady-state growth path.

The individual consumer maximizes a utility function, U_i, which has as arguments the levels of consumption in each of the two periods. 1/ Thus

$$(7) \quad U(x) = U(x_1, \ldots, x_N, x_{N+1}, \ldots, x_{2N}, L_1, L_2)$$

where x_i: $i \leq N$ refers to the i^{th} consumption good in period 1, x_i: $i > N$, refers to the i^{th} consumption good in period 2, and L_i refers to consumption of leisure in period i. Assume, again in order to be specific, that the utility function exhibits constant elasticity of substitution, and hence is of the form

$$(8) \quad U = \left(\sum_{i=1}^{2N} d_i^{1/u} x_i^{(u-1)/u} + \sum_{i=1}^{2} d_{2N+i}^{1/u} L_i^{(u-1)/u} \right)^{u/(u-1)}$$

where u is the consumer's elasticity of substitution and d_i: $i = 1, \ldots, 2N+2$ is the expenditure shares given to consumption goods, including leisure. Suppose that these expenditure shares reflect the consumer's rate of time preference, z, so that:

$$(9) \quad d_i/d_{i+N} = d_j/d_{j+N} = z: 1 \leq i, j \leq N$$

$$d_{2N+1}/d_{2N+2} = z$$

and, in addition, z is uniform across all consumers. None of these restrictions are essential to the working of the model, but they correspond to the normal macroeconomic interpretation of time preference. Hence leisure enters the utility function, but money, bonds, and capital do not.

The consumer maximizes his utility function subject to a set of intertemporal budget constraints, as we assume that capital markets are imperfect in that consumers cannot borrow against future income. 2/ The consumer must therefore cover his current expenditure plus savings from

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1/ There are K > 0 consumers in the model, however, in order to avoid unreadable subscripts the consumer demand parameters will not be indexed. It should be noted that these parameters, along with initial allocations are not uniform across consumers.

2/ This imperfection in capital markets is also the justification for not introducing separable utility functions.

current income. 1/ He has an initial allocation of money, M_0 at the beginning of period 1, and if he is a shareholder in the capital goods-producing firm he will also hold capital $\overline{K}_0$. There will, however, be no initial holdings of bonds. 2/ Let p_{Ki}, p_{Li}, p_{Mi}, p_{Bi} represent the prices of capital, labor, money, and bonds, respectively, in period i, and let TR_i represent whatever transfer payments the government pays to consumers during period i, while r_i represents this particular consumer's share in those transfers. 3/ The consumer's income, $I_1(p_1)$, valued at an arbitrary vector of prices $p_1 \equiv (p_{K1}, p_{L1}, p_{M1}, p_{B1})$ in period 1, is then given by

$$(10) \quad I_1(p_1) \equiv p_{M1}\overline{M}_0 + p_{K1}\overline{K}_0 + p_{L1}\overline{L}_0 + r_1 TR_1$$

In addition, the consumer has a second period budget constraint. If he has purchased a quantity, x_{B1}, of bonds in period 1, then receives the coupon value of those bonds in terms of units of money in period 2, valued at the price of money in period 1, thus being equal to $p_{M1}x_{B1}$, if we assume that the coupon payment is 1. 4/ The consumer's income in period 2, $I_2(p_1, p_2)$, then becomes

$$(11) \quad I_2(p_1, p_2) = p_{K2}\overline{K}_0 + p_{L2}\overline{L}_0 + p_{M1}x_{M1} + p_{M1}x_{B1} + r_2 TR_2$$

where x_{M1} is the quantity of money that the consumer holds in period 1. By supposing also that the bonds purchased in period 1 are long term, in the sense that they are not redeemable during the time span of the model, but continue to pay a uniform coupon, we are thus making an important assumption, namely, that there is no secondary market for capital, so that a consumer who holds capital cannot sell his capital to either other consumers or to enterprises. 5/ If we had a model with multiperiod

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1/ Another approach in, for example, Grandmont (1977) and Grandmont and Laroque (1975), is to have consumers borrow from the central bank against future income but to have no borrowing by the central bank. A number of technical problems are involved with allowing borrowing to go in both directions, essentially equivalent to the requirement of irreversibility of production.

2/ If the consumer held a quantity of bonds, $\overline{B}_0$, which came due in period 1 and were redeemed at par, then such an allocation would require us to impose an arbitrary assumption as to the rate of inflation from the past to period 1.

3/ The share could thus change from period to period.

4/ The rate of inflation is defined as $p_{M2}/p_{M1} - 1$, the percentage change in the price of money. Thus, an indexed bond would yield a coupon payment of p_{M2}, while a nonindexed bond, as we are considering, would pay p_{M1} in period 2.

5/ The interpretation of the price of capital in period i, p_{Ki}, is that it is a rental rather than a sales, or cost of production, price.

overlapping generations, then the possibility of one generation selling its capital to a new generation would have to be accounted for. Such a sale would take place when the rental stream of future earnings on the capital is discounted by the new generation over their life span by the future interest rate and is found to be at least equal to the sales price of capital. In a single generation model, however, the same discounting would be carried out by all consumers, so that no sales of capital would take place.

Although we do not explicitly solve for a third period, the model does have the notion of a future that is essentially the same as period 2. Thus the consumer will expect that the same relative prices will prevail in this future as existed in period 2, but that they will increase by whatever the expected rate of inflation is. The assumption then is that the consumer wishes to purchase the same bundle of consumption in the future as he purchased in period 2, subject to his rate of time preference. 1/ His expected future income, $I^E(p^E)$ is given by

$$(12) \quad I^E(p^E) \equiv p_L^E\bar{L}_0 + p_K^E\bar{K}_0 + p_{M2}x_{M2} + p_{M1}x_{B1} + p_{M2}x_{B2} + r_2 TR^E$$

where the superscript E denotes the expected value of the corresponding variable. In the third, or future period, the consumer will continue to receive the coupon payments from the bonds he purchased in period 1, valued at the price of money from that period, and he will also receive the corresponding interest payment on the bonds he has purchased in period 2. 2/

We are assuming that the consumer is myopic in the future, so that he expects no change in relative prices. Therefore we have, in particular, that

$$(13) \quad p_K^E = (1+\pi^E)p_{K2}, \; p_L^E = (1+\pi^E)p_{L2}, \; TR^E = (1+\pi^E)TR_2$$

where π^E is the expected rate of inflation. Although the method of derivation of π^E is not relevant to this study, one might continue to

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1/ As we mentioned earlier, this assumption is being made simply to close the model.

2/ As we shall see, the rental proceeds from the investment of period 1 will be fully exhausted as payments to bondholders in period 2, and the assumption of zero depreciation of capital, combined with perfect myopia in the future, will lead to the expected future (after period 2) rental income also being fully exhausted as payments to bondholders. The consumer does not distinguish between private and government bonds, each of which yields the same rate of return, and since returns to investment are fully exhausted as payments to bondholders, we need not specify ownership of newly produced capital.

assume myopia, so that π^E is equal to the actual rate of inflation between periods 1 and 2, inflation being defined as the rate of change in the price of money. 1/ Hence

$$(14) \quad \pi^E \equiv p_{M2}/p_{M1} - 1$$

The consumer, in solving his utility maximization problem, has three simultaneous budget constraints, one for each of the two perfectly anticipated periods, and one for the future period. Suppose that the consumer faces ad valorem taxes on his purchases of consumption goods, then let

$$(15) \quad t_1 \equiv (\tau_1, \ldots, \tau_N) \quad : 0 \leqq \tau_i$$

$$t_2 \equiv (\tau_{N+1}, \ldots, \tau_{2N}) : 0 \leqq \tau_i$$

where t_i represents the vector of tax rates levied on the N intermediate and final goods produced in period i. Let

$$(16) \quad \overline{p}_1 \equiv (\hat{p}_1, \ldots, \hat{p}_N), \; p_2 \equiv (\hat{p}_{N+1}, \ldots, \hat{p}_{2N})$$

denote the prices of the intermediate and final goods in each of the two periods. The value of the consumer's expenditure on all goods, including leisure and bonds, in period i is then given by

$$(17) \quad (1+t_i)\,\overline{p}_i \cdot x_i + p_{Li}\, x_{Li} + p_{Bi} x_{Bi}$$

where x_i, x_{Li}, x_{Bi} represent his consumption of goods, leisure, and bonds in period i, respectively. The consumer, in addition, requires a certain quantity of money to cover transaction costs. In Feltenstein (1983) this transaction demand is presented as a constant fraction of the value of consumption representing, in other words, a constant velocity of money.

Here we will present a somewhat more realistic version of the demand for money, in which demand for nominal cash balances depends not only on the value of current consumption but also on the nominal interest rate. Suppose we start with a simple quantity theory of money which, in our system, would be formulated as

$$(18) \quad p_{Mi} x_{Mi} = \frac{1}{v_i} (1+t_i)\overline{p}_i \cdot x_i$$

where v_i is the velocity of money in period i. Thus the nominal value of money demanded in period i is a function of the value of consumption of intermediate and final goods in that period. Leisure is not included as a determinant in the demand for money, since income taxes, as we shall discuss shortly, are withheld at the source, that is, the firm.

1/ It would be more correct to define inflation in terms of a consumer price index, rather than a one-commodity basket. Doing so would, however, require the introduction of index weights, which we wish to avoid.

Suppose now that v_i is not constant, but is a function of the nominal interest rate. The nominal interest rate, r_1, or the percentage return on a bond, in period 1 is given by

$$(19) \quad r_1 = (p_{M1} - p_{B1})/p_{B1}$$

while in period 2 it is given by

$$(20) \quad r_2 = (p_{M2} - p_{B2})/p_{B2}$$

since the bond pays a unitary coupon (in terms of units of money) in the period after which it is purchased. Suppose also that

$$(21) \quad \frac{1}{v_i} = ar_i^b: \; a \geqq 0, \; b \leqq 0$$

so that the velocity of money is directly related to the nominal interest rate. Hence, by substituting we arrive at

$$(22) \quad p_{M1}x_{M1} = a\left(\frac{p_{M1} - p_{B1}}{p_{B1}}\right)^b (1+t_1)\overline{p}_1 \cdot x_1$$

$$p_{M2}x_{M2} = a\left(\frac{p_{M2} - p_{B2}}{p_{B2}}\right)^b (1+t_2)\overline{p}_2 \cdot x_2$$

or, to put things in a somewhat more familiar form by taking logarithms,

$$(23) \quad \ln (p_{Mi}x_{Mi}) = \ln a + b \ln \left(\frac{p_{Mi} - p_{Bi}}{p_{Bi}}\right) + \ln(1+t_i)\overline{p}_i \cdot x_i$$

The total value of the consumer's period 1 and period 2 consumption must be equal to or less than the corresponding income, hence

$$(24) \quad (1+t_i)\overline{p}_i \cdot x_i + p_{Li}x_{Li} + p_{Bi}x_{Bi} + p_{Mi}x_{Mi} \leqq I_i(p_i)$$

where $p_{Mi}x_{Mi}$ is given by equation (22) and $I_i(p_i)$ is given by equations (10) and (11). For the third, or future, period we make the behavioral assumption that the consumer wishes to be able to purchase in the future the same bundle of consumption as he bought in period 2, discounted by his rate of time preference. 1/ Accordingly,

$$(25) \quad \frac{[(1+t_2)\overline{p}_2 \cdot x_2 + p_{L2}x_{L2}](1+\pi^E)}{1+z} = I^E(p^E)$$

1/ This assumption follows from the fact that his myopic expectations after period 2 cause him to anticipate the relative prices of period 2 to prevail in the future.

where $I^E(p^E)$ is defined in equation (12). We do not include money and bonds in equation (25), because to do so would presuppose an anticipated period beyond the third, or future, period.

We may consolidate equations (10), (11), (12), (22), (25) to obtain the following maximization problem for the consumer.

$$(26) \quad \max \left(\sum_{i=1}^{N} d_i^{1/u} x_i^{(u-1)/u} + \sum_{i=1}^{2} d_{2N+i}^{1/u} x_{Li}^{(u-1)/u} \right)^{u/(u-1)}$$

such that

$$(26a) \quad (1+t_1)\bar{p}_1 x_1 + p_{L1}x_{L1} + p_{B1}x_{B1} + a\left(\frac{p_{M2} - p_{B1}}{p_{B1}}\right)^b \bar{p}_1 x_1$$

$$\leqq p_{M1}\bar{M}_0 + p_{K1}\bar{K}_0 + p_{L1}\bar{L}_0 + r_1 TR_1$$

$$(26b) \quad (1+t_2)\bar{p}_2 x_2 + p_{L2}x_{L2} + p_{B2}x_{B2} + a\left(\frac{p_{M2} - p_{B2}}{p_{B2}}\right)^b (1+t_2)\bar{p}_2 x_2$$

$$\leqq p_{K2}\bar{K}_0 + p_{L2}\bar{L}_0 + p_{M1}x_{M1} + p_{M1}x_{B1} + r_2 TR_2$$

$$(26c) \quad \frac{(1+t_2)\bar{p}_2 x_2 + p_{L2}x_{L2}}{(1+z)} = p_{K2}\bar{K}_0 + p_{L2}\bar{L}_0 + \frac{p_{M1}x_{B1} + p_{M2}x_{M2} + p_{M2}x_{B2}}{(1+\pi^E)} + r_2 TR_2$$

After some algebraic manipulations, we may simplify equations (26a-c) to arrive at the following single budget constraint:

$$(27) \quad \left\{ \frac{p_{M1}}{p_{B1}}\left(1+a\left(\frac{p_{M1}-p_{B1}}{p_{B1}}\right)^b\right)\left(\frac{p_{B2}}{p_{M2}}+1\right) - a\left(\frac{p_{M1}-p_{B1}}{p_{B1}}\right)^b \right\} (1+t_1)\bar{p}_1 x_1$$

$$+ \left\{ 1 + \frac{p_{B2}}{p_{M2}(1+z)} + a\left(1-\frac{p_{B2}}{p_{M2}}\right)\left(\frac{p_{M2}-p_{B2}}{p_{B2}}\right)^b \right\} (1+t_2)\bar{p}_2 x_2$$

$$+ \left(\frac{p_{B2}p_{L1}}{p_{M2}p_{B1}} + \frac{p_{M1}p_{L1}}{p_{B1}}\right)x_{L1} + \left(\frac{p_{L2}+p_{B2}p_{L2}(1+\pi^E)}{p_{M2}(1+z)}\right)x_{L2}$$

$$\leqq \left(\frac{p_{B2}p_{M1}p_{K1}}{p_{M2}p_{B1}} + \frac{p_{B2}(1+\pi^E)}{p_{M2}} + p_{K2} + \frac{p_{M1}p_{K1}}{p_{B1}}\right)\bar{K}_0$$

$$+ \left(\frac{p_{B2}p_{L2}(1+\pi^E)}{p_{M2}} + \frac{p_{B2}p_{M1}p_{L1}}{p_{M2}p_{B1}} + p_{L2} + \frac{p_{M1}p_{L1}}{p_{B1}}\right)\bar{L}_0$$

$$+ \left(\frac{p_{B2}p_{M1}^2}{p_{M2}p_{B1}} + \frac{p_{M2}p_{M1}^2}{p_{B1}}\right)\bar{M}_0 + \left(\frac{p_{B2}p_{M1}}{p_{M2}p_{B1}} + \frac{p_{M1}}{p_{B1}}\right) r_1 TR_1$$

$$+ \left(\frac{p_{B2}(1+\pi^E)}{p_{M2}} + 1\right) r_2 TR_2$$

The derivation of the consumer's demands for individual goods is now straightforward. Define a new set of price vectors $\tilde{p}_1$, $\tilde{p}_2$, $\tilde{p}_{L1}$, $\tilde{p}_{L2}$ by

$$(28)\quad \tilde{p}_1 = \left\{ \frac{p_{M1}}{p_{B1}}\left(1+a\left(\frac{p_{M1}-p_{B1}}{p_{B1}}\right)^b\right)\left(\frac{p_{B2}}{p_{M2}}+1\right) - a\left(\frac{p_{M1}-p_{B1}}{p_{B1}}\right)^b \right\}(1+t_1)\bar{p}_1$$

$$\tilde{p}_2 \equiv \left\{ 1+\frac{p_{B2}}{p_{M2}(1+z)} + a\left(1-\frac{p_{B2}}{p_{M2}}\right)\left(\frac{p_{M2}-p_{B2}}{p_{B2}}\right)^b \right\}(1+t_2)\bar{p}_2$$

$$\tilde{p}_{L1} = \left(\frac{p_{B2}}{p_{M2}p_{B1}} + \frac{p_{M1}}{p_{B1}}\right)p_{L1}$$

$$\tilde{p}_{L2} = \left(1 + \frac{p_{B2}(1+\pi^E)}{p_{M2}(1+z)}\right)p_{L2}$$

In addition, let $W(p_1,p_2)$ denote the right-hand side of equation (27). It may then be shown that the consumer's demand for the j^{th} intermediate or final good is given by

$$(29)\quad x_j = \frac{W(p_1,\ p_2)}{\left[\sum_{i=1}^{2N} d_i\ \tilde{p}_i^{(1-u)} + \sum_{i=1}^{2} d_{2N+i}p_{Li}^{(1-u)}\right]}\ \frac{d_j}{\tilde{p}_j^u}$$

where $\tilde{p}_j$ represents the j^{th} component of $\tilde{p}^1$ if $j \leq N$, and the $j\text{-}N^{th}$ component of $\tilde{p}^2$ if $j > N$. 1/ The consumer's demand for leisure in periods 1 and 2, x_{Lj}, is given by

$$(30)\quad x_{Lj} = \frac{W(p_1,\ p_2)}{\left[\sum_{i=1}^{2N} d_i\ \tilde{p}_i^{(1-u)} + \sum_{i=1}^{2} d_{2N+i}p_{Li}^{(1-u)}\right]}\ \frac{d_{2N+j}}{p_{Lj}^u}$$

1/ To avoid confusing notations, we have not used the superscript K to denote the individual consumer.

The consumer's demand for money is derived from equation (22) as:

$$(31)\quad x_{M1} = \frac{a}{p_{M1}} \left[\frac{p_{M1} - p_{B1}}{p_{B1}}\right]^b (1+t_1)\overline{p}_1 \cdot x_1$$

$$x_{M2} = \frac{a}{p_{M2}} \left[\frac{p_{M2} - p_{B2}}{p_{B2}}\right]^b (1+t_2)\overline{p}_2 \cdot x_2$$

while the consumers demand for bonds, x_{Bi}, is given by

$$(32)\quad x_{B1} = \left\{ p_{M1}\overline{M}_0 + p_{K1}\overline{K}_0 + p_{L1}\overline{L}_0 + r_1 TR_1 - [1 + a(\frac{p_{M2} - p_{B1}}{p_{B1}})^b](1+t_1)\overline{p}_1 \cdot x_1 - p_{L1}x_{L1} \right\} / p_{B1}$$

$$x_{B2} = \left\{ \frac{[(1 + t_2)p_2x_2 + p_{L2}x_{L2}](1+\pi^E)}{1+z} - p_{L2}(1+\pi^E)\overline{L}_0 - p_{M1}x_{B1} - p_{K2}(1+\pi^E)\overline{K}_0 - a(\frac{p_{M2} - p_{B2}}{p_{B2}})^b (1+t_2)\overline{p}_2x_2 - r_2(1+\pi^E)TR_2 \right\} / p_{B2}$$

Having calculated the individual consumer's demands for all goods plus financial assets, we will now turn to the derivation of aggregate supply, and, accordingly, excess demand functions.

3. Financing the central government and the formation of capital

In our model there are two production activities that are not required to cover current costs, the production of public goods by the central government and the production of new capital by the investment activity. Consider the case of the central government first. In order to calculate the central government's financing requirements, and hence its emission of money and bonds, we must first derive its deficit (or surplus) in each period. This deficit depends, of course, upon the tax revenues that the government collects, which in turn, depend upon the level of supply. As before, let

$$p \equiv (p_1, p_2) = (p_{K1}, p_{L1}, p_{M1}, p_{B1}, p_{K2}, p_{L2}, p_{M2}, p_{B2})$$

be an arbitrary set of intertemporal prices for capital, labor, money, and bonds. Using the form of the individual industry's value-added functions, as given in equation (2), we obtain cost minimizing levels of use of capital for the j^{th} sector in period i:

$$(33)\quad K_j = (1+t_{Ki})^{(\alpha_j-1)} \left(\frac{1-\alpha_j}{\alpha_j}\right)^{(\alpha_j-1)} \left(\frac{p_{Ki}}{p_{Li}}\right)^{(\alpha_j-1)} Va_j : \begin{array}{l} i = 1 \text{ if } j \leqq N \\ i = 2 \text{ if } j > N \end{array}$$

where t_{Ki} and t_{Li} represent the tax rates levied on capital and labor, assumed to be uniform across sectors, in the i^{th} period, and Va_j represents the required inputs of value added, in real terms, to the j^{th} sector. 1/ We then derive L_j, the cost-minimizing inputs of labor to sector j, as

$$(34)\quad L_j = \left(\frac{1+t_{Ki}}{1+t_{Li}}\right)\left(\frac{1-\alpha_j}{\alpha_j}\right)\left(\frac{p_{Ki}}{p_{Li}}\right)K_j$$

and the nominal value added, va_j, is given by:

$$(35)\quad va_j(p) = p_{Ki}(1+t_{Ki})K_j + p_{Li}(1+t_{Li})L_j : \begin{array}{l} i = 1 \text{ if } j \leqq N \\ i = 2 \text{ if } J > N \end{array}$$

Given this vector va(p) of nominal value added, we may calculate intertemporal Leontief prices, $\bar{p}(p)$, as

$$(36)\quad \bar{p}(p) = va(p)(I-A)^{-1}$$

where A is the Leontief matrix of production defined in (1). We have thus calculated a set of 2N prices that give zero profit to each activity operating in each period, corresponding to the assumed prices for capital and labor. A complete set of intertemporal prices is now given for all intermediate and final goods, as well as capital, labor, and financial assets, so the consumer's maximization problem may be solved as in equations (29)-(32). In particular, we may derive total demand for the j^{th} intermediate and final good, xL_j, as 2/

$$(38)\quad xL_j \equiv \sum_{k=1}^{K} x_j^k$$

where x_j^k is the k^{th} consumer's demand for intermediate or final good j, as in equation (29), and where the summation is taken over all K consumers. 3/

1/ The interpretation of t_{Ki} is a profit tax levied upon capital, while t_{Li} may be thought of as an income tax that is collected at the source, that is, a withholding tax.

2/ Here xL_j is supposed to denote a Leontief good.

3/ In equation (29) we did not use the superscript k to denote the individual consumer in order to avoid confusing notation.

We may then derive the vector of activity levels, z, of the 2N activities required to produce this level of demand as

$$(39) \quad z = (I-A)^{-1}\, xL$$

Let y_{Kj}, y_{Lj} be the requirements of column j for capital and labor, as derived in equations (33) and (34). The total requirements for capital and labor by private industry in periods 1 and 2 are then

$$(40) \quad y_{KP1} = \sum_{j=1}^{N} z_j\, y_{Kj}, \quad y_{LP1} = \sum_{j=1}^{N} z_j\, y_{Lj}$$

$$y_{KP2} = \sum_{j=N+1}^{2N} z_j\, y_{Kj}, \quad y_{LP2} = \sum_{j=N+1}^{2N} z_j\, y_{Lj}$$

The total taxes collected by the central government in each of the two periods may now be calculated. If T_i denotes the taxes collected in period i, then

$$(41) \quad T_1 = \sum_{j=1}^{N} t_j\, xL_j + t_{K1} y_{KP1} + t_{L1}\, y_{LP1}$$

$$T_2 = \sum_{j=N+1}^{2N} t_j\, xL_j + t_{K2} y_{KP1} + t_{L2}\, y_{LP1}$$

In addition, the government also uses capital and labor to produce public goods in each of the two periods. Suppose that the real quantity of these public goods is given by Q_1, Q_2. 1/ The government has a Cobb-Douglas production function as given in equation (6), and we may derive the cost-minimizing quantities of capital and labor, y_{KGi}, y_{LGi}

used by the government in producing $\overline{Q}_i$, and the total cost to the government, G_i of producing this quantity.

$$(42) \quad y_{KGi} = \left(\frac{(1-\beta_i)}{\beta_i}\right)^{(\beta_i-1)} \left(\frac{p_{Ki}}{p_{Li}}\right)^{(\beta_i-1)} \overline{Q}_i$$

$$y_{LGi} = \frac{(1-\beta_i)}{\beta_i}\, \frac{p_{Ki}}{p_{Li}}\, y_{KGi}$$

$$G_i = p_{Ki}\, y_{KGi} + p_{Li}\, y_{LGi}$$

1/ We are supposing that the government sets expenditure targets for public goods in real terms, irrespective of the cost of inputs.

The deficit of the central government in period 1, D_1, is then given by

$$(43) \quad D_1 \equiv G_1 - T_1$$

so that if D_1 is negative the government runs a surplus. In the case of a surplus. Assume that the surplus is paid out as transfer payments to consumers, 1/ but in the case of a deficit, financing operations must take place. In Feltenstein (1983) the assumption is made that the value of bond financing is a constant fraction of the value of the government deficit. Here, however, we will suppose that the quantity of bonds issued is a continuous function of the deficit of the following form:

$$(44) \quad y_{BG1} = h_1\left(\frac{D_1}{p_{B1}}\right)\frac{D_1}{p_{B1}}: \; 0 \leqq h_1\left(\frac{D_1}{p_{B1}}\right) \leqq 1, \; h_1' > 0$$

Here h_1 is a continuous function of D_1/p_{B1} and represents the fraction of the government's deficit financed by the sale of bonds in period 1, while y_{BG1} represents the government's sale of bonds in period 1. Thus as the price of bonds falls, $h_1' > 0$ implies that the government will finance an increasingly greater proportion of its deficit via the sale of bonds. 2/ The remainder of the financing of the deficit will come from the issuance of money, $\bar{y}_{M1}$, where

$$(45) \quad \bar{y}_{M1} = \left[1 - h\left(\frac{D_1}{p_{B1}}\right)\right]\frac{D_1}{p_{M1}}$$

In period 2 the formation of the government deficit is somewhat different, since it must pay not only for its current consumption but also for its debt obligations incurred in period 1. Accordingly,

$$(46) \quad D_2 = G_2 + p_{M1} \cdot y_{BG1} - T_2$$

1/ These transfer payments are not identically equal to the sum of the transfer payments, included in the consumers' budget constraints, although at equilibrium they will be.

2/ Intuitively, the government can finance a real deficit via money creation equal only to the real value of the money the public is willing to hold. An examination of equations (28) and (29) indicates, however, that by lowering the price of period 1 bonds the government can cause the public to forgo current consumption in favor of future consumption, thereby lowering the requirements for period 1 capital and labor as inputs to private production. In this case there are more real resources available for the government, so that, in particular, a greater portion of its deficit can be financed by bonds.

The issuance of money and bonds in period 2 may be defined as before, using a new function h_2 such that

(47) $$y_{BG2} = h_2\left(\frac{D_2}{P_{B2}}\right) \cdot \frac{D_2}{P_{B2}}: \quad 0 \leqq h_2\left(\frac{D_2}{P_{B2}}\right) \leqq 1, \; h_2' < 0$$

$$y_{M2} = \left[1 - h_2\left(\frac{D_2}{P_{B2}}\right)\right] \frac{D_2}{P_{M2}}$$

where h_2 is continuous. We would, of course, have $h_1 = h_2$ if the government chooses to maintain the same financing rule in period 2 as in period 1.

Let us now define x_{Mi}, x_{Bi}, aggregate consumer demand for money and bonds in period i, as

(48) $$x_{Mi} = \sum_{j=1}^{K} x_{Mi}^{j}, \quad x_{Bi} = \sum_{j=1}^{K} x_{Bi}^{j}$$

where x_{Mi}^{j}, x_{Bi}^{j} are as defined in equations (31) and (32). 1/ Consider then $x_{Bi} - y_{BGi}$, the excess demand for bonds, over and above the amount that is being supplied by the government. If $x_{Bi} - y_{BGi} < 0$, then bonds are in excess supply. In this case we will set $y_{Bpi} = 0$, where y_{Bpi} is the amount of bonds issued by the private investment activity in period i. Thus, since investment must be financed by sales of bonds, there will be no investment in period i. Suppose now that $x_{Bi} - y_{BGi} \geqq 0$. In this case there is an excess demand for bonds, that is, savings, which must be satisfied by the issuance of private bonds. At the same time, we have as an equilibrium condition that the rental payments on new capital must be the same as the bond payments corresponding to this capital. Accordingly,

(49) $$p_{K2} H_1 = p_{M1}(x_{B1} - y_{BG1})$$

(50) $$p_{K2}(1+\pi^E)H_2 = p_{M2}(x_{B2} - y_{BG2})$$

where H_i is the real quantity of capital produced in period i. 2/

We thus have

(51) $$H_1 = \frac{p_{M1}(x_{B1} - y_{BG1})}{p_{K2}}: \quad x_{B1} - y_{BG1} \geqq 0$$

$$H_2 = \frac{p_{M2}(x_{B2} - y_{BG2})}{p_{K2}(1+\pi^E)}: \quad x_{B2} - y_{BG2} \geqq 0$$

1/ As before, j now refers to the individual consumer.

2/ Recall that investment in period i does not yield a return until period i+1.

Equation (51) may be interpreted as giving the levels of real investment in each of the two periods, corresponding to the equilibrium conditions for rental payments on investment. It is straightforward to show that the cost minimizing levels of use of capital and labor in period i, corresponding to a level of investment H_i, are

$$(52) \quad y_{KHi} = [b_{1i}^{1/s_i} + b_{2i}^{(s_i-1)/2s_i} \, b_{1i}^{(1-s_i)/s_i} \, (p_{Li}/p_{Ki})^{(s_i-1)\, s_i/(1-s_i)}]^{(s_i-1)/s_i} \, H_i$$

$$(53) \quad y_{LHi} = \frac{b_{2i}}{b_{1i}} \left(\frac{p_{Li}}{p_{Ki}}\right)^{s_i} K_i$$

where the terms in equation (52) are as those defined in the investment function given in equation (3).

The cost of producing this level of investment must be covered by the sale of bonds, which are sold in period i at a discount price p_{Bi}. We then must have

$$(54) \quad p_{Bi} \, y_{BPi} = p_{Ki} y_{KHi} + p_{Li} y_{LHi}$$

where y_{BPi} is the quantity of privately issued bonds in period i, and y_{KHi}, y_{LHi} have been derived in equations (52) and (53). Since p_{Bi} is given, the required private issue of bonds is y_{BPi}, where

$$(55) \quad y_{BPi} = \frac{p_{Ki} y_{KHi} + p_{Li} y_{LHi}}{p_{Bi}}$$

We would have that normally $y_{BPi} \neq x_{Bi} - y_{BGi}$. 1/ We then define the total supply of bonds in period i, y_{Bi}, as

$$(56) \quad y_{Bi} \equiv y_{BGi} + y_{Bpi}$$

We have now described the financing of government deficits and investment, and, via equation (51), have derived a savings-driven formation of capital. It is intuitive that consumers have a demand for savings that must be satisfied by the purchase of bonds. Since there is no risk, government and private bonds are identical, from the consumer's point of view. We are thus implicitly assuming that government bonds appear on the market before privately issued bonds, so that if their supply is sufficient to satisfy the demand for bonds, no private investment takes place. 2/

1/ It follows that the discount being offered on period i bonds is not necessarily that which is required to make the equality hold.

2/ Another, perhaps more plausible, interpretation of our method of deriving private capital formation, suggested by Kenneth Rogoff, is that

4. Excess demand functions and the conditions for intertemporal equilibrium

In order to demonstrate the existence of an intertemporal equilibrium, that is, a set of prices for all goods and financial assets at which markets clear in both periods, we must first derive excess demand functions. The approach that we use is based upon Feltenstein (1983), in which endogenous supplies of money and bonds, along with an endogenous government deficit, are reflected by extra dimensions in the price simplex and, accordingly, additional elements in the excess demand function. Here, the fact that behavior in one period of the model is related to behavior in the other period, combined with the presence of the issuance of private debt, makes the construction of these excess demand functions somewhat more complicated. 1/

The presence of an intertemporal input-output matrix allows us to confine the vector of excess demand functions to the space of prices corresponding to capital, labor, money, bonds, and transfer payments, indexed by their time period. Accordingly, given an arbitrary vector of prices p, we may derive the nominal value added per unit of output for each of the 2N sectors producing intermediate and final goods, as in equation (35). Equation (36) then gives Leontief prices for each of the two periods, and equations (38) and (39) give total demand for intermediate and final goods, along with the corresponding level of production required of each activity in the Leontief matrix. Equations (40)-(42) derive the total required inputs of capital and labor in each period by the private sector and the government, so we may derive the aggregate supplies of capital and labor in period i, $\bar{y}_{Ki}$, $\bar{y}_{Li}$, by the government and that part of the private sector producing intermediate and final goods as

(57) $\bar{y}_{K1} = y_{KP1} + y_{KG1}$, $\bar{y}_{L1} = y_{LP1} + y_{LG1}$

The total requirements of capital and labor in each period include also their usage in investment. Equations (43) and (46) calculate the government deficit or surplus and lead to the real value of investment in period i, H_i, as in equation (51). 2/ Equations (52) and (53)

2/ (continued from p. 19) the government, unlike the private sector, will not decrease its borrowing as interest rates rise but will continue to attempt to finance the real deficit that it incurs, irrespective of how low it drives the price of bonds.

1/ In the earlier paper, there was also an endogenous balance of payments, which we are ignoring here because of our assumption of a closed economy.

2/ Recall that we are not allowing for depreciation, so that no distinction is made between net and gross investment.

calculate the inputs of capital and labor, y_{KHi}, y_{LHi}, required by the investment activity in period i, so the corresponding total requirements, $\hat{y}_{Ki}$, $\hat{y}_{Li}$, are given by

$$(58) \quad \hat{y}_{Ki} = y_{KHi} + \tilde{y}_{Ki}, \ y_{Li} = y_{LHi} + \tilde{y}_{Li}$$

The total supplies of the capital and labor, y_{Ki}, y_{Li}, are then

$$(59) \quad y_{K1} = -\hat{y}_{K1} + \overline{K}_0, \ y_{K2} = -\hat{y}_{K2} + \overline{K}_0 + H_1$$

$$y_{L1} = -\hat{y}_{L1} + \overline{L}_0, \ y_{L2} = -\hat{y}_{L2} + \overline{L}_0$$

where $\overline{K}_0$, $\overline{L}_0$ are the aggregate initial stocks of capital and labor, summed over all consumers, and H_1 is the real level of investment in period 1.

The change in the money supply in period i, $\hat{y}_{Mi}$, is given by equations (45) and (47), so that the total supply of money in each period, y_{Mi}, is

$$(60) \quad y_{M1} = \overline{M}_0 + \hat{y}_{M1}, \ y_{M2} = y_{M1} + \hat{y}_{M2}$$

The supply of bonds in each period, y_{Bi}, is given by

$$(61) \quad y_{Bi} = y_{BGi} + y_{Bpi}$$

where y_{BG1}, the government's issuance of bonds in period 1, is given by equation (44), y_{BG2} is given by equation (47), and y_{Bpi} is derived from equation (55).

We have now derived an aggregate supply vector, y, where

$$(62) \quad y \equiv (y_1, y_2) \equiv (y_{K1}, y_{L1}, y_{M1}, y_{B1}, y_{K2}, y_{L2}, y_{M2}, y_{B2})$$

As in Feltenstein (1983), we augment this supply vector by two additional dimensions, corresponding to transfer payments in each of the two time periods. Accordingly, define y(p), the augmented supply vector, by

$$(63) \quad y(p) \equiv (y, \delta(D_1), \delta(D_2)): \quad \begin{array}{l} \delta(D_i) = D: D \leqq 0 \\ \delta(D_i) = 0: D > 0 \end{array}$$

where D_i, the government deficit in period i, is given by equations (43) and (46), and y is given by equation (62).

The derivation of an augmented demand vector, x(p), is now straightforward. Consumer demand for capital is zero, hence 1/

$$(64) \quad x_{Ki} \equiv 0: \quad i = 1, 2$$

1/ This follows from the fact that consumers are assumed to satisfy their demand for savings entirely through the purchase of bonds.

Equations (29) through (32) give individual demands for leisure, money, and bonds in each period, so adding across consumers gives the aggregate demands, x_{Li}, x_{Mi}, x_{Bi}. The aggregate demand vector, x, is then defined by

(65) $x \equiv (x_1, x_2) \equiv (0, x_{L1}, x_{M1}, x_{B1}, 0, x_{L2}, x_{M2}, x_{B2})$

Finally, the augmented demand vector, x(p), is defined by

(66) $x(p) \equiv (x, -TR_1, -TR_2)$

where TR_i represents the proxy for government transfer payments that enters the consumer's maximization problem, as given in equation (26). The aggregate excess demand function, u(p), is then defined as

(67) $u(p) \equiv x(p) - y(p)$

so we must show that there exists some price p*, where

(68) $p^* = (p^*_{K1}, p^*_{L1}, p^*_{M1}, p^*_{B1}, p^*_{K2}, p^*_{L2}, p^*_{M2}, p^*_{B2}, TR^*_1, TR^*_2)$

such that $u(p^*) \leq 0$, that is, such that supply is equal to or greater than demand and that transfer payments received by consumers are equal to or greater than the amount actually paid out. 1/

Certain restrictions must be placed upon the behavior of the government in order to ensure the existence of an equilibrium. The essence of these restrictions is intuitively plausible, namely, that the government's program of public spending be not so great as to require more capital than the initial stock of the economy plus the maximum that can be produced by first period investment. Such restriction would, of course, always be valid in any empirical application of the model. The equilibrium condition of equation (68) may be calculated by a method that will be discussed briefly in the next section; it will, in particular, yield the effect of the government's spending on the level of output and investment of the private sector, the goal of this study.

III. Conclusion

We have constructed a computational general equilibrium model that is designed to analyze the crowding out of the private sector by the public sector. The model is fully dynamic, having two periods plus a past (before the first period) and a future (after the second period). Both consumers and firms have perfect foresight for the two periods in question, so that prices, tax liabilities, and transfers received from the government in period 2 are correctly forecast in period 1. Private enterprises are constrained to cover current expenditures from current revenues,

1/ A proof of this result is available from the author.

while investment is financed by the sale of bonds. The government, on the other hand, is not required to cover the cost of its production of public goods, the quantity of which it sets in real terms. When it incurs a deficit, the government issues a combination of money and bonds to cover its loss, but a surplus, if it occurs, is distributed to consumers. Perfect foresight precludes the possibility of risk, so that private and public bonds are perfect substitutes, from the point of view of the consumer. The equilibrium conditions on public and private debt are quite different, however, since capital produced by private investment in period 1, and hence coming on line in period 2, must yield a return in period 2 equal to the debt obligations on the bonds that financed it. The government, on the other hand, must add the debt obligations incurred in period 1 and coming due in period 2 to its current expenditures in that period.

The model represents a considerable advance over the two general types of model that have been used to analyze crowding out. Macroeconomic and monetary models have been used to examine financial crowding out, when government borrowing causes the interest rate to rise, thereby making credit more expensive to the private sector and forcing it to curtail its level of investment. These models have, however, a number of simplifications that preclude the type of analysis that we are able to make. They do not, for example, disaggregate the private sector and therefore are unable to allow for differential impact of interest rate changes on firms that have large investment programs versus those that do not. Perhaps more important, they do not distinguish between individual taxes and therefore cannot consider the effects of changes in single taxes, as our model is able to do. The models that are able to cope with these problems of aggregation, namely, computational general equilibrium systems, have had a number of inherent weaknesses of their own. They do not admit the existence of financial assets and therefore require a balanced government budget, an unrealistic assumption that our model, containing both money and bonds, is not forced to make. Because of the absence of money, these models also lack any notion of a price level, so that questions concerning, for example, indexation of taxation or debt cannot be considered. Such questions are, however, easily admissible within our system. Finally, because there are no bonds in these models, they cannot analyze private borrowing to finance investment; hence they must totally ignore the problem of financial crowding out. Our model, containing both public and private borrowing competing for private savings, can be used to address precisely this problem.

There are a number of directions for future research. First and most obvious would be the empirical implementation of our model. The basis for the required computer program was developed in Feltenstein (1983) and should permit modification to solve this system. Fullerton and others (1981) have constructed the real side of a computational general equilibrium model, applied to the United States, while Jorgenson (1983) has estimated value-added functions of the form that our model requires on an industry-by-industry basis. A number of studies have estimated investment functions for the United States, 1/ so as a first step we could use the results, although eventually we might wish to carry out our own estimations. The main task, then, would be to estimate the financial side of the model, assuming that we wish to apply it to the United States.

This estimation is not so overwhelming as it seems, since if we estimate a nominal-demand-for-money function and combine it with the estimates of Fullerton and others (1981) of consumer demand for goods, the demand for bonds, equivalent to savings in our model, can be derived as a residual. The fully estimated model might then be simulated to analyze, for example, the extent to which the crowding out of the private sector is affected by different rates of consumer time preference, a parameter that we cannot directly estimate. The policymaker interested in the trade-off between financing it via increased public indebtedness may, as a secondary concern, wish to examine the impact that his choice of policies has upon interest rates and private output, variables that will be generated by our model. If it is felt that the degree of crowding out that the model predicts is excessive, the possibility of cutting the deficit by raising individual tax rates could also be studied, and simultaneously the impact that the new tax regime would have on the welfare of different consumer groups could be calculated. There are, of course, many other policy simulations that could be carried out with an empirical version of the model.

On the theoretical side, the model could be extended to include a foreign trade sector with an endogenous balance of payments, allowing for foreign borrowing to finance private investment and government deficits, but if we wish to incorporate a floating exchange rate it would be necessary to develop a theory of exchange rate determination within our framework. Such a theory would require a notion of risk and portfolio decision in exchange markets and would represent a considerable advance over our current system. 2/ On the domestic side, an important innovation would be to introduce an overlapping generations structure to the model. Because the old and the new generations would be discounting the future stream of returns on capital over different time horizons, such a structure would permit the existence of a secondary market for capital. Finally, we have avoided allowing capital gains taxation for certain technical reasons. A significant improvement in the model would be achieved by introducing such taxes, since the consumer's attitude toward bond purchases would then be directly affected by the tax regime.

1/ See, for example, Jorgenson and Stephenson (1969), Christensen and Jorgenson (1970), and Jorgenson (1971) and (1974).

2/ Feltenstein (1982) considers a system of managed floating but avoids the issue of risk.

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