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WP/96/118

INTERNATIONAL MONETARY FUND

Western Hemisphere Department

Macroeconomic Fluctuations and Equilibrium Discount Factors

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October 1996

Abstract

The estimation of discount factors is a central issue in empirical finance, particularly in the literature on excess volatility. In particular, it is difficult to find empirical discount factors that are volatile enough to account for fluctuations in asset prices. This paper constructs discount factors from some macroeconomic time series commonly used in empirical models of asset prices. Data for the U.S. stock market imply some evidence that discount factors relate to macroeconomic conditions, but comparison of the estimated discount factors to Hansen-Jagannathan (1991) bounds shows that the candidate discount factors cannot account for the volatility in asset returns.

JEL Classification Numbers:

G10, G12

1/ I thank Steven Dunaway, Robert Hauswald, Thomas Helbling, and John McDermott for helpful comments and discussions.

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Summary

The characterization of discount factors is central to finance theory and practice. Different models of asset returns are distinguished by their specifications for discount factors: in Lucas's (1978) model, for example, the discount factor is the intertemporal marginal rate of substitution. The empirical challenge for these models is to produce a discount factor that is volatile enough to match the low volatility of fundamentals (e.g., dividends) with the high volatility of prices. This challenge is of more than academic interest, since it lies at the center of the debate over "excess volatility" in financial markets.

However, meeting this challenge has proved difficult. The innovative econometric technique devised by Hansen (1982) and Hansen and Singleton (1982) has led to much work in fitting intertemporal consumption-based models to consumption and returns data. Simple versions of this model have been unable to produce discount factors that can explain the equity premium or the risk-free rate (Hansen and Jagannathan, 1991).

Along largely separate lines, much empirical work has appealed to various macroeconomic time series other than consumption as proxies for the driving variables that determine stock returns and engender risk premia. This paper explores whether some macroeconomic time series commonly used in financial applications can be used to construct a discount factor for an intertemporal model of stock returns. Both linear and nonlinear models for the discount factor are examined. The models are generally significant in a statistical sense and show some evidence of a nonlinear relationship between discount factors and macroeconomic time series, but the fitted models for discount factors are not volatile enough to match the equity premium or the treasury-bill yield. Hence, it is not clear how models built from such factors can account for the volatility in asset returns.

I. Introduction

Much research in empirical finance has been devoted to finding plausible discount factors (also referred to as pricing kernels) for asset payoffs. 1/ The search for discount factors is of much more than academic interest; indeed, the debate over 'excess volatility' in asset returns hinges on whether variation in expected returns can explain the variation in observed returns. Models of asset prices generally engender a relationship similar to

$$\text{Price} = f(\text{payoffs, discount factors}).$$

The discount factors tell us how to translate the random distribution of payoffs into a value of a claim to that random distribution. In economic models, the discount factors themselves may be random variables. In empirical work, since actual payoffs (such as dividends) and prices are observable, and since (assuming for example rational expectations and ergodicity) the distribution of payoffs can be inferred from their empirical distribution, the real job in assessing the consistency of payoffs and prices is estimating the discount factors.

However, there are many ways to calculate discount factors--one way for each model of asset pricing. Moreover, empirical results can be sensitive to the way that discount factors are calculated. For example, in the line of inquiry often associated with Shiller (1989), discount factors are often taken to be constant. 2/ Hence, much of the research that finds 'excess volatility' can be alternatively characterized as finding that discount factors are not constant (that expected returns are not constant), so that the issue of excess volatility hinges crucially on whether plausible discount factors can be variable enough to account for the variability in asset returns. 3/

Indeed, the characterization of discount factors is central to finance theory and practice. Different models of asset returns are distinguished by their specification for discount factors: in the intertemporal model of Lucas (1978), for example, the discount factor is the intertemporal marginal rate of substitution. The main empirical challenge for these models is to produce a discount factor that is volatile enough to match the low volatility of fundamentals (e.g., dividends) with the high volatility of prices. However, meeting this challenge has proved difficult. The innovative econometric technique of Hansen (1982) and Hansen and Singleton (1982) has led to much work in fitting intertemporal consumption-based

1/ Here by 'discount factors' I mean something related to but more general than the discount factor in the typical discounted-dividend model. This issue is discussed in more detail later in the paper.

2/ Notable exceptions in that literature include McDermott (1994) and Flood, Hodrick, and Kaplan (1994), which use discount factors based on intertemporal optimization.

3/ See, e.g., Kleidon (1988).

models to consumption and returns data. Simple versions of these models (e.g., based on time-separable power utility) have been unable to produce discount factors that are sufficiently volatile to match stock returns and in particular have been unable to produce discount factors that can explain the high premium for stocks over Treasury bills (the equity premium puzzle) and the low return on Treasury bills per se (the risk-free rate puzzle). ^{1/} This failure to produce sufficient volatility in discount factors without imposing an unrealistic degree of risk aversion and/or prudence is pointed up by the results of diagnostic tests based on Hansen and Jagannathan's (1991) volatility bounds for discount factors.

Along largely separate lines, much empirical work has appealed to various macroeconomic time series other than consumption as proxies for the driving variables that determine stock returns and engender risk premia. ^{2/} In this approach, stock returns are usually described by a linear function of variables such as inflation, various bond yields, and cyclical indicators. While this approach has proved useful in understanding the cross-sectional structure of asset returns and, indeed, has seen more practical application than the intertemporal approach, it has largely existed apart from the literature that seeks to explicitly characterize discount factors. ^{3/}

In this paper, I explore whether some macroeconomic time series commonly used in financial applications can be used to construct a discount factor for an intertemporal model of stock returns. Both linear and nonlinear models for the discount factor are examined. The models generally are significant in a statistical sense and show some evidence for a nonlinear relationship between discount factors and macroeconomic time series, but the fitted models for discount factors are not nearly volatile enough to match the equity premium or the Treasury-bill yield. Hence, it is not clear how models built from such factors can account for the volatility in asset returns.

^{1/} See Mehra and Prescott (1985). There is some evidence to suggest that habit formation, loss aversion, nonexpected utility, or incomplete markets could account for these puzzles; see Constantinides (1990), Campbell and Cochrane (1995), Benartzi and Thaler (1995), Epstein and Zin (1991), and Weil (1992). For a skeptical note on these solutions to the puzzle see Kocherlakota (1996).

^{2/} The literature on this topic is large, but representative papers include Chen, Roll and Ross (1986), Ferson and Harvey (1991), and Fama and French (1993). A brief survey is in Fama (1991).

^{3/} Some notable exceptions are Bansal and Viswanathan (1993), and Bansal, Hsieh, and Viswanathan (1993), who relate discount factors to nonlinear functions of asset returns. Harvey and Kirby (1995) discuss strategies for estimating similar models for discount factors using GMM. Agoro-Menyang (1995) estimates time-series models for latent discount factors in asset returns. Also, in a somewhat different but related vein of inquiry, Campbell (1993) has advocated using macroeconomic time series as a proxy for consumption data; see also Campbell (1992).

II. Discount Factors and Stock Returns

Equilibrium in frictionless markets, or the absence of arbitrage, yields a precise relationship between the current price of an asset, P_t , and its next-period payoff (price plus dividend), $P_{t+1} + D_{t+1}$:

$$P_t = E_t [m_{t+1} (P_{t+1} + D_{t+1})] \quad 1/ \quad (1)$$

where E_t denotes conditional expectation. One example familiar from the macroeconomics literature is the Euler equation from a infinitely-lived representative agent model in discrete time (Lucas (1978);

$$U'(C_t) P_t = E_t [\beta U'(C_{t+1}) (P_{t+1} + D_{t+1})],$$

where $U(\cdot)$ denotes the one-period utility function, β an impatience parameter, and C_t current period consumption. Dividing both sides by $U'(C_t)$ gives the discount factor $m_{t+1} = \beta U'(C_{t+1})/U'(C_t)$. This expression emphasizes the dependence of the stochastic discount factor m_{t+1} on preferences (e.g. time preference and risk aversion).

If m_t is constant over time and equal to \bar{m} , then

$$P_t = \bar{m} E_t [(P_{t+1} + D_{t+1})].$$

If \bar{m} is less than one in absolute value, we can iterate this expression forward (provided that dividends do not grow explosively) and use iterated expectations to yield

$$P_t = \sum_{i=1}^{\infty} \bar{m}^i E_t D_{t+i}$$

This model for discount factors has been used widely in tests for market efficiency. 2/ The fact that (as is evident from the Euler equation version of the discount factor) the discount factor, and hence expected returns, need not be constant over time has led some to criticize that approach and advocate estimating discount factors using the consumption-based approach. 3/ However, the intertemporal consumption-based approach to specifying discount factors has had some difficulty in reconciling the data with standard specifications of utility functions, in particular reasonable with respect to assumptions regarding the degree of risk aversion and/or prudence. 4/ In a separate vein, there has been much econometric work on specifying asset returns as a function of measurable macroeconomic variables in the context of linear factor models, factor-ARCH models, and the like. 5/ I combine these literatures by explicitly specifying the discount factor m as a function of macroeconomic variables that have proved

1/ See Duffie (1992), Chapters 1 and 2.

2/ See Shiller (1989).

3/ See, e.g. McDermott (1994).

4/ See Kocherlakota (1996).

5/ See Fama (1991).

useful in other contexts for explaining the behavior of asset returns. The approach followed here is to fit candidate models for m_{t+1} to the data using the Generalized Method of Moments (GMM). 1/ This method affords a direct way to check whether the state variables are good proxies for the time-variation in risk premia, and whether the economic model that uses such proxies can explain the longer-term variation in the dividend/price ratio. The models are then checked by standard GMM hypothesis tests and by comparing the properties of the implied discount factor (the fitted value of m) to the properties that are implied by stock-market data, as in Hansen and Jagannathan (1991).

The estimation strategy is as follows. Suppose that the model for m_t can be written as $m_t = g(x_t|\theta)$, where θ represents the model's parameters and x_t is a vector of macroeconomic variables. For example, assuming that the conditional joint distribution of m_t and x_t is elliptical would yield a linear model for $g(\cdot|\cdot)$. 2/ Rewriting the Euler/no-arbitrage equation (1) in terms of $R_t = (P_{t+1} + D_{t+1})/P_t$ and substituting the model for m gives

$$1 = E_t [g(x_{t+1}|\theta)R_{t+1}]. \quad (2)$$

This formulation yields orthogonality conditions that can be exploited to derive estimates of θ via GMM, as well as test statistics for the statistical significance of the estimates of elements of θ and a specification test for the orthogonality conditions.

The parameter estimates can also be used to construct a time series for the estimated candidate discount factor $m_t = g(x_t|\theta)$. The properties of such a candidate discount factor can then be checked against the empirical bounds derived by Hansen and Jagannathan (1991). 3/ The derivation and computation of these bounds is discussed in the Appendix.

III. Models for Discount Factors

As emphasized above, stochastic discount factors may relate to the characteristics of preferences (impatience and risk aversion). 4/ Hence one might expect stochastic discount factors to relate to macroeconomic time series whose fluctuations could reflect movements in preferences. In

1/ See Hamilton (1994), Chapter 14.

2/ See Harvey and Kirby (1995).

3/ Specifically, the bounds used are those derived assuming no riskless payoff and imposing their Restriction 1 (see Hansen and Jagannathan (1991), pp. 233-5).

4/ They may (rather than will) relate to preferences as stochastic discount factors will exist in the absence of arbitrage; there need not exist a supporting equilibrium (see Duffie (1992), Chapter 2). Alternatively, one may think of certain macroeconomic time series as serving as sufficient statistics for the information in the discount factor, as in Harvey and Kirby (1995).

addition, if tastes and technology vary over the business cycle, one might expect stochastic discount factors to vary along with time series that reflect the state of the business cycle. Time series that meet these criteria were chosen as a basis for the models of stochastic discount factors. The 4-time series were PI, the consumer price inflation rate; DEF, the default premium for corporate bonds over Treasury bonds of a similar maturity; RTB, the real yield on a three-month Treasury bill (nominal yield minus CPI inflation); and YC, the slope of the yield curve (the thirty-year Treasury bond rate less the three-month Treasury bill rate). ^{1/}

Return series are also required in order to estimate the discount-factor model in equation (2). Since two leading puzzles in asset pricing are the equity premium and the risk-free rate, the returns series used are the equity premium (the excess of the return on the Standard and Poor's 500 index over the three-month Treasury bill return) and the real three-month Treasury bill rate. The term "risk-free rate" is used to describe this series, since this is standard in finance. It should be noted, however, that the ex-post real rate actually used in the tests is not riskless, though the Hansen-Jagannathan diagnostics applied later do not assume the existence of an ex-ante risk-free real rate of return.

Data on consumption are purposely ignored in this exercise. There is not much evidence that consumption growth (as opposed to the intertemporal marginal rate of substitution) is significantly related to stock returns. ^{2/} Asset-market series like the ones used here are measured with much less error than consumption data and might actually serve as a better proxy for consumption services or the evolution of aggregate wealth; indeed, several recent studies find that log-linearizing the budget constraint in order to replace consumption data with asset-market data can yield better estimates of dynamic asset-pricing models. ^{3/} Finally, the point of this exercise is to assess whether the sorts of time series commonly used in empirical studies of asset pricing can be used to construct a discount factor. Since outside a strict consumption-based model framework, few of these studies find consumption data useful, it is not employed here.

Five models for discount factors are fitted: a linear model,

$$m_t = a + b_1 PI_t + b_2 DEF_t + b_3 RTB_t + b_4 YC_t; \quad (3)$$

a quadratic model,

$$m_t = a + b_1 PI_t + b_2 DEF_t + b_3 RTB_t + b_4 YC_t \\ + c_1 PI_t^2 + c_2 DEF_t^2 + c_3 RTB_t^2 + c_4 YC_t^2; \quad (4)$$

^{1/} Details on the data are provided in Table 1.

^{2/} See Mankiw and Shapiro (1986).

^{3/} See Campbell (1992), Campbell (1993), and Hardouvelis, Kim, and Wizman (1995).

a linear model augmented with the absolute differences of the macro variables from their means (to proxy for volatility following Granger and Ding (1993)),

$$m_t = a + b_1 PI_t + b_2 DEF_t + b_3 RTB_t + b_4 YC_t \\ + c_1 \text{abs}(PI_t - \text{mean}(PI)) + c_2 \text{abs}(DEF_t - \text{mean}(DEF)) + c_3 \text{abs}(RTB_t - \text{mean}(RTB)) \\ + c_4 \text{abs}(YC_t - \text{mean}(YC)); \quad (5)$$

a linear model augmented with absolute changes in the macro variables (again to proxy for volatility),

$$m_t = a + b_1 PI_t + b_2 DEF_t + b_3 RTB_t + b_4 YC_t \\ + c_1 \text{abs}(\Delta PI_t) + c_2 \text{abs}(\Delta DEF_t) + c_3 \text{abs}(\Delta RTB_t) + c_4 \text{abs}(\Delta YC_t); \quad (6)$$

and a linear model augmented with Generalized Autoregressive Conditional Heteroskedasticity in Mean (GARCH-M) estimates of volatility,

$$m_t = a + b_1 PI_t + b_2 DEF_t + b_3 RTB_t + b_4 YC_t \\ + c_1 \sigma_t^2(PI) + c_2 \sigma_t^2(DEF) + c_3 \sigma_t^2(RTB) + c_4 \sigma_t^2(YC), \quad (7)$$

where $\sigma_t^2(\cdot)$ denotes a GARCH estimate of conditional variance at date t .

These models correspond closely to models used in the literature for stock returns themselves. The linear model is closely related to arbitrage pricing or linear factor models. 1/ The polynomial model can capture nonlinear relationships between the macroeconomy and asset markets. 2/ The two volatility-related models correspond to a rich literature that explores the dynamics in conditional volatility in asset markets and its relationship to equilibrium returns. 3/ The models are fitted by substituting the proposed functional forms into equation (2) above and employing GMM to estimate the model parameters. The instruments used were lags of each of the four variable driving the discount factor plus a constant. Four instrument lags were used for each model, except for the model involving GARCH volatilities. Since the GARCH volatilities themselves contained up to four lags of information, eight lags were used for that model. This meant 82 orthogonality conditions for the GARCH model and 44 orthogonality conditions for the others (1 + number of instruments x number of lags) x (number of asset-return series).

1/ See Chen, Roll, and Ross (1986) and Epps and Kramer (1996).

2/ See Hiemstra and Kramer (1994), Bansal and Viswanathan (1993), and Bansal, Hsieh, and Viswanathan (1993).

3/ See Bollerslev, Chou, and Kroner (1992) for a discussion of ARCH and Factor-ARCH models and Granger and Ding (1993) for a discussion of the absolute value as a measure of volatility.

IV. Estimation Results for Discount-Factor Models

Table 2 shows the estimated parameters, t-statistics, and the p-value of the GMM test of the overidentifying restrictions (J-statistic) for the model. In the linear model (Panel A), three of the four macroeconomic variables have a statistically significant relationship to the discount factor. A 1-percent increase in inflation tends to increase the discount factor by about 0.05 percent, while increases in the default premium and real Treasury bill yield decrease the discount factor by about 0.05 and 0.1 percent, respectively. 1/ The yield curve has a statistically insignificant and very small coefficient, and so is probably not important. Since discount factors are inversely related to expected returns, it makes sense that a higher default premium or a higher rate of return on a particular asset is consistent with a higher rate of return on assets in general (e.g., a smaller discount factor). The negative relationship between inflation and expected returns accords with both theory and past empirical studies. 2/

Panel B of Table 2 shows results for the quadratic representation of the discount factor. 3/ This version could be thought of either as a more sophisticated approximation to the true (nonlinear) function relating the discount factor to macroeconomic variables, or as adding terms for macroeconomic risk to the linear representation. When the squares of the variables are added, the coefficients in the linear part generally lose statistical significance (except for the real Treasury bill yield) and in two cases reverse sign as well. Inflation squared and the real Treasury bill yield squared both have statistically significant coefficients, and what is probably the appropriate sign--namely that an increase in the squared value (e.g., risk) decreases the discount factor and increases expected return. The squared default premium and squared yield curve have no statistically discernable effect on the discount factor, and the yield curve squared has a positive rather than negative sign.

The results with the polynomial discount factor suggest that macroeconomic volatility might be an important determinant of the discount factor (though squared variables might not be the best specification for volatility). Hence, the last three results are for models that add measures of volatility risk to the linear representation.

For the model with the absolute value of the difference from the mean as a measure of risk (Panel C), the estimates for the linear part are again

1/ The data, including the stock returns, are all in units where 0.01 implies a 1-percent monthly rate of return (as would be indicated by the equilibrium relationship (2)).

2/ See, e.g., Stulz (1986).

3/ A version of the model that included cross-product terms produced some statistically significant coefficients, and a highly variable fitted value for the discount factor, but the coefficients were so large in absolute value as to be implausible.

of mixed significance. The coefficient for inflation, while statistically significant, is positive as in the strictly linear model, while the other linear components other than the real Treasury bill yield are not statistically significant. Inflation volatility is significant, and has a sensible (negative) sign, but the only other significant volatility variable (the volatility of the real Treasury bill yield) has a positive sign. The results using the absolute value of the change as a measure of volatility (Panel D) are likewise mixed.

The model with GARCH volatilities (Panel E) does not fare much better. 1/ In a statistical sense, this model for volatility fits the data well--all the linear coefficients are statistically significant, as are all but one of the coefficients on conditional volatility. However, the positive coefficient on the yield curve is hard to interpret, as are the positive signs of the volatility coefficients (not to mention the very large coefficient on the volatility of the yield curve).

Overall, the results are mixed. The models have significant coefficients for the linear part, and in many cases for the nonlinear parts, but the signs and magnitudes of the coefficients are in some cases hard to interpret. One issue that the parameter estimates do not address is whether the estimated discount factors are sufficiently volatile to explain the volatility in returns. Another diagnostic for these models, normally applied to parametric estimates based on utility functions, is the mean-variance bound for discount factors derived by Hansen and Jagannathan (1991). This diagnostic is applied in the next section.

V. Can the Discount Factors Explain the Volatility in Stock Returns?

The models fit fairly well in a statistical sense, but the more substantive economic issue is whether the models generate discount factors that are volatile enough to explain the equity risk premium and the risk-free rate. This issue is addressed by comparing the mean and variance of each candidate discount factor to the region of means and variances consistent with the stock-return data, as in Hansen and Jagannathan (1991). Briefly, this region, derived from stock returns alone, is a lower bound on the standard deviation of an admissible discount factor for a given expected value of the discount factor. 2/

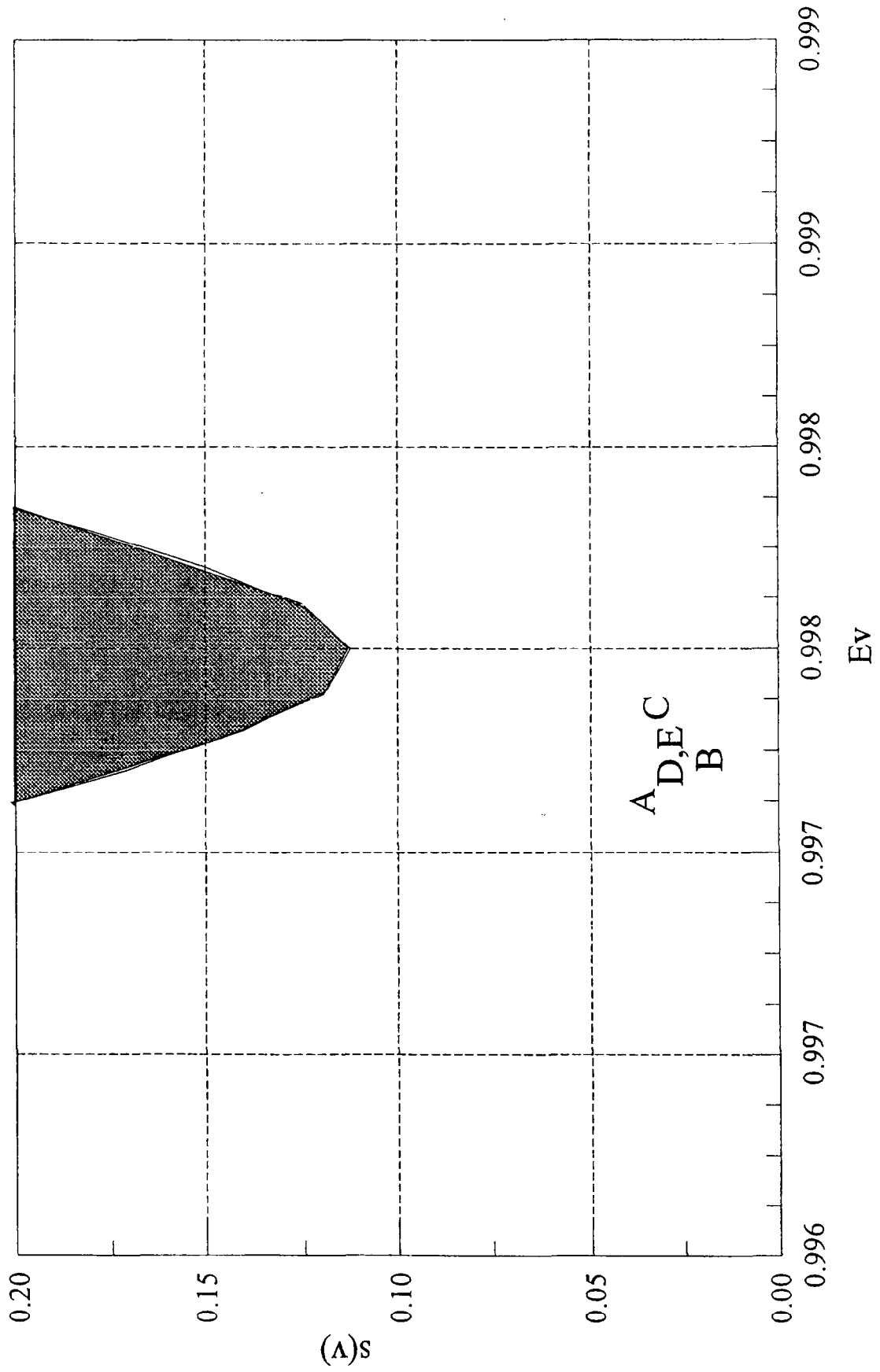
Chart 12 shows the admissible region based on the data for the equity premium and risk free rate used in the tests as the shaded area above the parabola. 3/ It is clear that none of the candidate discount factors is

1/ A version where m depended on conditional standard deviations rather than conditional variances yielded qualitatively similar results.

2/ The Hansen-Jagannathan bounds are discussed in more detail in the Appendix.

3/ Means and standard deviations for the fitted discount factors are also shown in Table 2.

Chart 1. Hansen-Jagannathan Bounds for Discount Factors 1/



1/ Shaded region is the set of admissible means and standard deviations for discount factors. Letters correspond to actual means and standard deviations of discount factors described in Table 2.

nearly volatile enough to explain the data. In fact, the standard deviations of these discount factors are about a factor of 10 too small to fit the data. Worse still, the candidate discount factors cannot even outperform those derived from the standard consumption-based model. It appears that the candidates fitted in the previous section fall short of explaining the volatility in stock and bond returns.

VI. Conclusions

This paper constructed candidate stochastic discount factors from four macroeconomic time series that are commonly used in empirical asset-pricing models. Five candidate functions for discount factors, ranging from a linear function to a function based on GARCH volatilities, were estimated using GMM. Several facts come through clearly in this analysis. First, these factors seem to be significant determinants of the stochastic discount factor. In many cases the coefficients for the variables, including terms measuring risk, showed up significantly in the estimated equations. Second, as shown by the Hansen-Jagannathan bounds, simple functions like those employed here do not generate nearly enough volatility from those factors to match the moments of asset returns. It is conceivable that there is some function that could do this, though it is not clear that a specification search would necessarily tell us much. ^{1/}

These findings give one pause in considering the recent research in applied finance that uses macroeconomic variables like the ones used here. The equation examined is quite general and does not depend on a specific model for utility or choice. A variety of functions of the variables failed to generate discount factors that were sufficiently volatile to match the moments of asset returns. Hence, it seems hard to imagine that these same variables are good proxies for systematic risk in arbitrage pricing models. Whether there is some function that can generate a useful stochastic discount factor from these variables or similar ones awaits future research.

^{1/} Some experiments with neural network functions, for example, were unsuccessful in identifying even fairly simple semi-nonparametric functions of the four series as in Bansal and Viswanathan (1993).

The Hansen-Jagannathan (1991) Bounds for Discount Factors

There is a rich literature in financial economics on mean-variance portfolio efficiency. 1/ This literature focuses on the efficient portfolio frontier, or the set of portfolios which minimize the standard deviation of return for a given mean return. Hansen and Jagannathan (1991) recently extended this literature by deriving an efficient frontier for the stochastic discount factor or pricing kernel in equilibrium or no-arbitrage models of asset prices. In their scheme, asset data imply a particular mean-variance efficient frontier for eligible discount factors; that is, for a given mean of the discount factor, their formula give a lower bound for the standard deviation of the discount factor. If the discount factor is not at least that volatile, given its mean, it is insufficiently volatile to match asset returns. This approach is appealing precisely because one fundamental problem in modeling asset returns has been finding sensible discount factors that are sufficiently volatile to match asset returns with fundamentals. 2/

One derivation of these bounds is as follows. 3/ The notation follows Hansen and Jagannathan (1991) while the development follows Cochrane and Hansen (1992). Let q denote the vector of asset prices today, h denote the vector of asset payoffs in a subsequent period, and m denote the true discount factor. For example, in (1) in the text, q would equal P_t , and h would equal $P_t + 1 + D_t + 1$. No-arbitrage or equilibrium, plus iterated expectations, implies

$$E(q) = E(hm). \quad (A1)$$

Denote $E(m)$ by ν . Consider a linear regression of m onto $(h-E(h))$,

$$m = \nu + (h-E(h))'\beta + e \quad (A2)$$

The regression slopes in β are given by

$$\beta = \Sigma^{-1} \text{Cov}(h,m) \quad (A3)$$

where Σ is the covariance matrix of h . (A1) implies that

$$\text{Cov}(h,m) = E(q) - \nu E(h), \quad (A4)$$

and substituting this into (A3) gives

$$\beta = \Sigma^{-1} (E(q) - \nu E(h)). \quad (A5)$$

1/ See Huang and Litzenberger (1988), Chapter 3.

2/ For example, the so-called 'excess volatility/predictability' literature can be seen as a confirmation that discount rates (expected returns) are not constant over time; see e.g., Kleidon (1988).

3/ I skip many details here, including strict no-arbitrage restrictions that impose $m > 0$. See Hansen and Jagannathan (1991).

That is, given ν , the regression coefficients in (A2) can be calculated from asset payoffs (h) and prices (q).

Since e and h are uncorrelated by construction, the regression (A2) gives a decomposition of the variance of m into two parts:

$$\sigma^2(m) = \sigma^2((h - E(h))'\beta) + \sigma^2(e). \quad (A6)$$

Substituting (A5) yields

$$\sigma^2(m) = [(E(q) - \nu E(h))'\Sigma^{-1} (E(q) - \nu E(h))] + \sigma^2(e). \quad (A7)$$

or the standard-deviation bound

$$\sigma(m) \geq [(E(q) - \nu E(h))'\Sigma^{-1} (E(q) - \nu E(h))]^{1/2}. \quad (A8)$$

That is, for a specific ν , (A8) gives a minimum variance that the corresponding discount factor must have, given asset market data for h and q. Extensions to this framework deal with estimation error (Cochrane and Hansen (1992)), restrictions on higher moments (Snow (1991)), and its relationship to tests of mean-variance spanning (Bekaert and Urias (1996)).

GARCH estimation results

Table A1 displays the results of GARCH-M (Generalized Autoregressive Conditional Heteroskedasticity-in-Mean) estimation of conditional volatilities for each of the four macroeconomic time series. ^{1/} Estimation of the GARCH models was preceded by a regression of squared values of each time series on its lagged squared values and tests for ARCH in each series in order to assess the statistical importance of time-dependence in volatility. Each of the regressions showed evidence of some such dependence, and each of the ARCH tests rejected the null hypothesis of no first-order ARCH dependence at the one percent level. The next step was to fit to each series the GARCH-M model

$$y_t = c + \theta \sigma_t + e_t, \quad (A9)$$

$$e_t \sim N(0, \sigma_t^2) \quad (A10)$$

$$\sigma_t^2 = \alpha_0 + \sum_i \alpha_i e_{t-i}^2 + \sum_i \beta_i \sigma_{t-i}^2 \quad (A11)$$

where y denotes the series of interest (PI, DEF, YC, or RTB); σ_t^2 denotes its volatility, and e_t is a serially-independent shock to y. The model relates the level of y to its volatility, and its volatility to past volatility and squared shocks to the level equation. Such models have been found to provide a good description of a number of macroeconomic time series,

^{1/} See Hamilton (1994), Chapter 21, for a discussion of models of autoregressive heteroskedasticity.

including inflation and various asset yields (see Bollerslev, Chou and Kroner (1992)).

The models were first estimated with four autoregressive lags and four moving-average lags. These lags and the mean component (represented by θ) were then generally removed when t-tests indicated that their coefficients were statistically insignificant. The exception was the default factor (DEF), where the insignificant ARCH coefficient was retained as preliminary results clearly indicated the presence of first-order ARCH dynamics. Fitted values from these models were then used in the construction of the GARCH discount factor.

Table A1: GARCH Results Used in GARCH Model of Stochastic Discount Factor

GARCH model:

$$y_t = c + \theta \sigma_t + e_t,$$

$$e_t \sim N(0, \sigma_t^2)$$

$$\sigma_t^2 = \alpha_0 + \sum_i \alpha_i e_{t-i}^2 + \sum_i \beta_i \sigma_{t-i}^2$$

	Parameter	Estimate	T-Statistic
<u>Model for PI</u>			
	α_0	0.0013	3.91
	α_1	0.3349	1.69
<u>Model for DEF 1/</u>			
	α_0	0.0000	0.05
	α_1	0.9408	0.57
<u>Model for RTB</u>			
	θ	0.9122	10.27
	α_0	0.0000	1.23
	α_1	0.1242	4.25
	β_1	0.8377	23.40
<u>Model for YC</u>			
	α_0	0.0001	1.39
	β_1	0.8779	9.38

1/ While the estimated coefficients were not significant, the fitted values for $\sigma_t(\text{DEF})$ were used in constructing the GARCH discount factor as preliminary regressions using squared values showed evidence of ARCH.

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Table 1. Data Series Used for the Model

(Sample Period: January 1984--April 1996)

Series	Mnemonic	Source
Stock Market Return (used to compute Hansen-Jagannathan bounds for discount factors; Standard and Poor's 500)	...	Standard and Poor's
Default Premium (Aaa-rated corporate yield less 30-year Treasury bond yield)	DEF	Moody's / Federal Reserve
Inflation (percent change in CPI, all urban consumers)	PI	Department of Labor
Real Treasury Bill Yield (3-month Treasury bill yield less CPI inflation)	RTB	Federal Reserve / Department of Labor
Yield Curve (30-year Treasury bond yield less 3-month Treasury bill yield)	YC	Federal Reserve

Table 2. Candidate Stochastic Discount Factors (Continued)

A. Linear model: P-value = 0.73

$$\hat{m} = a + b_1 \text{PI} + b_2 \text{DEF} + b_3 \text{RTB} + b_4 \text{YC}$$

0.999	0.046	-0.062	-0.092	-0.001
(3845.91)	(12.41)	(-3.19)	(-24.21)	-(0.22)

Descriptive statistics for \hat{m} :

Mean: 0.9972 Standard deviation: 0.0036

B. Polynomial (quadratic) model: P-value = 0.73

$$\hat{m} = a + b_1 \text{PI} + b_2 \text{DEF} + b_3 \text{RTB} + b_4 \text{YC}$$

0.995	-0.001	0.074	-0.044	-0.047
(186.28)	(-0.07)	(0.93)	(-2.04)	(-0.67)

$$+ c_1 \text{PI}^2 + c_2 \text{DEF}^2 + c_3 \text{RTB}^2 + c_4 \text{YC}^2$$

-0.508	-20.552	-0.382	+ 1.34
(-2.193)	(-0.801)	(-1.849)	(0.840)

Descriptive statistics for \hat{m} :

Mean: 0.9975 Standard deviation: 0.0021

C. Absolute value of difference from mean (volatility) model:
P-value = 0.74

$$\hat{m} = a + b_1 \text{PI} + b_2 \text{DEF} + b_3 \text{RTB} + b_4 \text{YC}$$

0.998	0.023	0.138	-0.102	-0.011
(1425.87)	(2.178)	(1.473)	(-9.886)	(-1.379)

$$+ c_1 \text{abs}(\text{PI} - \text{mean}(\text{PI})) + c_2 \text{abs}(\text{DEF} - \text{mean}(\text{DEF})) + c_3 \text{abs}(\text{RTB} - \text{mean}(\text{RTB}))$$

-0.087	-0.129	0.037
(-5.102)	(-0.714)	(2.239)

$$+ c_4 \text{abs}(\text{YC} - \text{mean}(\text{YC}))$$

0.044
(1.221)

Descriptive statistics for \hat{m} :

Mean: 0.9978 Standard deviation: 0.0029

Table 2. Candidate Stochastic Discount Factors (Concluded)

D. Absolute value of changes (volatility) model: P-value = 0.51

$$\begin{aligned} \hat{m} = & a + b_1 \text{PI} + b_2 \text{DEF} + b_3 \text{RTB} + b_4 \text{YC} \\ & 0.999 + 0.052 - 0.059 - 0.099 - 0.013 \\ & (2687.23) (9.053) (-1.543) (-14.558) (-1.643) \\ & + c_1 \text{abs}(\Delta \text{PI}) + c_2 \text{abs}(\Delta \text{DEF}) + c_3 \text{abs}(\Delta \text{RTB}) + c_4 \text{abs}(\Delta \text{YC}) \\ & - 0.026 0.251 - 0.002 0.103 \\ & (-1.009) (0.700) (-0.092) (3.143) \end{aligned}$$

Descriptive statistics for \hat{m} :

Mean: 0.9973 Standard deviation: 0.0039

E. GARCH volatility model: 1/ P-value = 0.99

$$\begin{aligned} \hat{m} = & a + b_1 \text{PI} + b_2 \text{DEF} + b_3 \text{RTB} + b_4 \text{YC} \\ & 0.976 + 0.042 - 0.145 - 0.091 + 0.016 \\ & (403.915) (13.688) (-3.614) (-27.873) (4.399) \\ & + c_1 \sigma^2(\text{PI}) + c_2 \sigma^2(\text{DEF}) + c_3 \sigma^2(\text{RTB}) + c_4 \sigma^2(\text{YC}) \\ & + 0.355 + 2.133 + 1.341 + 32.914 \\ & (6.809) (1.478) (4.412) (9.619) \end{aligned}$$

Descriptive statistics for \hat{m} :

Mean: 0.9973 Standard deviation: 0.0035

1/ Coefficients estimated by GMM using equation (2): $1 = E_t[g(x_{t+1}|\theta)R_{t+1}]$. T-statistics are based on heteroskedasticity- and autocorrelation-consistent standard errors (with 4 moving-average lags). The instruments used were 4 lags of each variable plus a constant.

1/ See the Appendix for the estimation results for the GARCH models for each series.